

A PARALLEL TO THE NULL IDEAL FOR INACCESSIBLE λ

PART I

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ABSTRACT. It is well known to generalize the meagre ideal replacing \aleph_0 by a (regular) cardinal $\lambda > \aleph_0$ and requiring the ideal to be λ^+ -complete. But can we generalize the null ideal? In terms of forcing, this means finding a forcing notion similar to the random real forcing, replacing \aleph_0 by λ , so requiring it to be $(< \lambda)$ -complete. Of course, we would welcome additional properties generalizing the ones of the random real forcing. Returning to the ideal (instead of forcing) we may look at the Boolean Algebra of λ -Borel sets modulo the ideal. Surprisingly we get a positive = existence answer for λ a “mild” large cardinal: the weakly compact one. We apply this to get consistency results on cardinal invariants for such λ 's. We shall deal with other cardinals more properties related forcing notions in a continuation.

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[We deal with a generalization.]

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[Here we give the simplest way to prove $(\text{CON}(\mathfrak{b}_\lambda > \text{cov}_\lambda(\text{meagre})))$; recall [Sh:945] solves Matet problem $\text{CON}(\mathfrak{d}_\lambda > \text{cov}_\lambda(\text{meagre}))$; λ strongly inaccessible.]

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[We try to deal with the large cardinal issues in §(4B); start with smaller and/or with larger.]

§ 0. INTRODUCTION

§ 0(A). **Aims: for general audience.**

The ideals on the reals of null sets and of meagre sets are certainly central in mathematics. From the forcing point of view we speak of random real forcing and Cohen forcing. The Cohen forcing has natural generalizations (and relatives) when we replace $\mathcal{P}(\mathbb{N})$ by $\mathcal{P}(\lambda)$, or the set of the characteristic functions of subsets of λ for λ regular uncountable replacing finite by cardinality $< \lambda$, but we lack a generalization of random real forcing to higher cardinals λ , replacing reals by λ -reals, e.g. members of ${}^\lambda 2$. It has seemed that this lack is due to nature; the reason being that on the one hand the Baire category theorem generalizes naturally (when we are allowed to approximate in λ -steps and information of size $< \lambda$ instead finite), but on the other hand we know nothing remotely like Lebesgue measure.

Surprisingly, at least for me, there is a generalization: not of the Lebesgue measure, but of the ideal of null sets, i.e. the ones of Lebesgue measure zero. This is done here (i.e. in this part) for a mild large cardinal λ : weakly compact. The solution for more cardinals will be dealt with in a continuation. The present definition should be examined in two ways. First, we may list the well known properties of the null ideal (and of random real forcing) and try to prove (or disprove) them for our ideal. Second, random real forcing was used quite extensively in independence results; in particular, related cardinal invariants so it is natural to generalize those uses. The first issue is dealt with in §3 (assuming Definition 1.3) and intended for wider audience. The second is treated in §4. Whereas success in the second issue should be easy to judge, concerning the first issue the reader may first list what are reasonable hopes and compare them with the discussion and description in the beginning of §3 and more in §(3A), §(3B), this is not done in §(0A) in order to help the reader make a list of expectations independent of what is done.

A set theoretically uninitiated reader may read the rest of §(3A) to see what are those large cardinals, look casually at Definition 1.3, just enough to see that the definition of \mathbb{Q}_κ , the parallel of the set of closed subset of $[0, 1]_\mathbb{R}$ or ${}^\omega 2$ of which are not null (= the forcing \mathbb{Q}_κ for κ strongly inaccessible is natural and simple, then jump to §3 (up to §(3C)) to see what we hope and what is done.

Let us describe for the non-set-theoretic reader, what are these “large cardinals”. Note that \aleph_1 is parallel in some respect to \aleph_0 , whereas \aleph_0 is “the first infinite cardinal”; the number of natural numbers; and \aleph_1 is the first uncountable cardinals, and is the number of countable ordinals (that is, isomorphism types of countable linear well orderings). Also both are so called regular: the union of less than \aleph_ℓ sets each of cardinality $< \aleph_\ell$ is $< \aleph_\ell$. But \aleph_0 is strong limit: $\kappa < \aleph_0 \Rightarrow 2^\kappa < \aleph_0$ whereas \aleph_1 is not. We can prove that there are strong limit cardinals: let $\beth_0 = \aleph_0$, $\beth_{n+1} = 2^{\beth_n}$, $\beth_\omega = \sum_n \beth_n$, now \beth_ω is a strong limit cardinal but alas is not regular. We say a cardinal λ is (strongly) inaccessible when λ is regular and strong limit, it is called “large cardinal” because we cannot prove its existence in ZFC but, modulo this, it is considered a very reasonable one. Similarly, the weakly compact ones which we now introduce. We may require more on λ : the analog of the infinite Ramsey theorem; every graph with λ nodes has a subgraph with λ nodes which is complete or empty. So weakly compact cardinals are more similar to \aleph_0 than other cardinals, so it is not unnatural assumption trying to generalize the null ideal.

* * *

§ 0(B). For Set Theorists.

Here we prove that for λ weakly compact there are forcing notions adding a new $\eta \in {}^\lambda 2$ which have not few parallel (replacing “finite” by “of cardinality $< \lambda$ ”) of the properties associated with random real forcing, (and define the relevant ideals). It seems natural to hope this will enable us to understand better related problems, in particular cardinal invariants of λ ; on cardinal invariants for $\lambda = \aleph_0$, i.e. the continuum see Blass [Bla]; in higher cases see Cummings-Shelah [CuSh:541]; in particular on strongly inaccessible see Roslanowski-Shelah [RoSh:777], [RoSh:888], [RoSh:889], [RoSh:942] and also [Sh:945].

Concerning λ -cardinal invariants we deal here with one such a problem: $\lambda < \text{cov}_\lambda(\text{meagre}) < \mathfrak{b}_\lambda < \mathfrak{d}_\lambda$, we also deal with $\prod_{\varepsilon < \lambda} \theta_\varepsilon$ having D_ε an $|\varepsilon|^+$ -complete filter on $\theta_\varepsilon, \theta_\varepsilon < \lambda$, but more systematic treatment is delayed.

In §1 we show for λ weakly compact that there is a (non-trivial) λ -bounding λ^+ -c.c. ($< \lambda$)-strategically complete forcing notion and even a λ -complete one, see 0.4 (and then with having $\theta_\varepsilon, D_\varepsilon$).

In §2 we deal with adding many subsets to λ close to measure product, with full presentation. Easily there are some parameters, in particular, filters. In §(4A) we give the trivial choice of those parameters and compute $\text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$. By this,

starting with supercompact, we get $\text{CON}(\lambda \text{ strongly inaccessible}, 2^\lambda \geq \mathfrak{d}_\lambda = \mathfrak{b}_\lambda > \text{cov}(\text{meagre}) > \lambda)$. In §(4B) we deal with starting with smaller (than supercompact) large cardinal and/or ending with large cardinals.

In §3 we try to deal systematically with parallels of properties of the null ideal.

In part II we shall continue, in particular we shall deal with eliminating the assumption “ λ weakly compact”, and also with starting not with Cohen but other nice forcing notions and more.

§ 0(C). Preliminaries.

Definition 0.1. 0) We say η is a λ -real when $\eta \in {}^\lambda 2$.

1) We define when $\mathbf{B} \subseteq {}^\lambda 2$ is a λ -Borel set naturally, that is (see [Sh:630]) $X \subseteq {}^\lambda 2$ is a basic λ -Borel set if there exists $\nu \in {}^{\lambda > 2} 2$ such that $X = ({}^\lambda 2)^{[\nu]} = \{\eta \in {}^\lambda 2 : \nu \triangleleft \eta\}$. The family of λ -Borel sets is the closure of the basic ones under unions and intersections of at most λ members, and complements.

2) “ F is a λ -Borel function” is defined similarly.

3) $\mathbf{B} \subseteq {}^\lambda 2$ is a $\Sigma_1^1(\lambda)$ -set when $\mathbf{B} = \{\langle \eta(2\alpha) : \alpha < \lambda \rangle : \eta \in \mathbf{B}_1\}$ for some λ -Borel \mathbf{B}_1 .

4) $B \subseteq {}^\lambda 2$ is a λ -stationary Borel set when for some λ -Borel function $F : {}^\lambda 2 \rightarrow \mathcal{P}(\lambda), \eta \in B \Leftrightarrow F(\eta)$ is stationary.

5) Similarly replacing ${}^{\lambda > 2} 2$ by other trees with λ levels and λ nodes.

Definition 0.2. 1) We say $B \subseteq {}^\lambda 2$ is λ -closed when:

- $\eta \in {}^\lambda 2 \wedge (\forall \alpha < \lambda)(\exists \nu \in B)(\eta \upharpoonright \alpha = \nu \upharpoonright \alpha) \Rightarrow \eta \in B$
equivalently

- for some tree $T \subseteq {}^{\lambda>2}$ we have $B = \lim_{\lambda}(T) = \{\eta : \eta \text{ a sequence of length } \lambda \text{ such that } \alpha < \lambda \Rightarrow \eta \upharpoonright \alpha \in T\}$.

- 2) We say $B \subseteq {}^{\lambda>2}$ is a \mathbb{Q} -basic set when $B = \lim_{\lambda}(p)$ for some $p \in \mathbb{Q}$ assuming \mathbb{Q} is a family of subtrees of ${}^{\lambda>2}$ (or a quasi order with such set of elements).
- 3) Similarly replacing ${}^{\lambda>2}$ by other trees.

Definition 0.3. 1) We say that a forcing notion \mathbb{P} is α -strategically complete when for each $p \in \mathbb{P}$ in the following game $\partial_{\alpha}(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.

A play lasts α moves; in the β -th move, first the player COM chooses $p_{\beta} \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_{\beta}$ and $\gamma < \beta \Rightarrow q_{\gamma} \leq_{\mathbb{P}} p_{\beta}$ and second the player INC chooses $q_{\beta} \in \mathbb{P}$ such that $p_{\beta} \leq_{\mathbb{P}} q_{\beta}$.

The player COM wins a play if he has a legal move for every $\beta < \alpha$.

- 2) We say that a forcing notion \mathbb{P} is $(< \lambda)$ -strategically complete when it is α -strategically complete for every $\alpha < \lambda$.

Remark 0.4. The difference between “ \mathbb{P} is λ -strategically complete” and “ λ -complete” is not real, i.e. when we do not distinguish between equivalent forcing, those properties are the same (as in [Sh:f, Ch.XIV]).

Definition 0.5. 1) The λ -Cohen forcing is $({}^{\lambda>2}, \triangleleft)$.

- 2) A forcing notion \mathbb{Q} is λ -bounded or ${}^{\lambda}\lambda$ -bounded when $\Vdash_{\mathbb{Q}}$ “for every function f from λ to λ there is $g \in ({}^{\lambda}\lambda)^{\mathbf{V}}$ such that $f \leq g$, i.e. $\alpha < \lambda \Rightarrow f(\alpha) \leq g(\alpha)$ ”.

- 3) We say a \mathbb{Q} -name $\eta \in {}^{\alpha}\beta$ is a generic of \mathbb{Q} when for some sequence $\langle \tau_p : p \in \mathbb{Q} \rangle$, τ_p a function definable in \mathbf{V} (or even a $(|\alpha| + |\beta|)$ -Borel one) from ${}^{\alpha}\beta$ into $\{0, 1\}$ we have $\Vdash “p \in \dot{\mathbf{G}} \text{ iff } \tau_p(\eta) = 1”$.

Definition 0.6. 1) We say “ S is nowhere stationary” when S is a set of ordinals, and for every ordinal δ of uncountable cofinality, $S \cap \delta$ is not a stationary subset of δ .

- 2) For a set p of sequences of ordinals and η let $p^{[\eta]} = \{\nu \in p : \nu \leq \eta \text{ or } \eta \leq \nu\}$.

Definition 0.7. We say $h : \lambda \rightarrow \mathcal{H}(\lambda)$ is a Laver diamond when for every $\chi > \lambda$ and $x \in \mathcal{H}(\chi)$ there is a normal ultrafilter D on $[\mathcal{H}(\chi)]^{<\lambda}$ such that $x = \mathbf{j}_1(\mathbf{j}_D(h)(\lambda))$ when:

- (a) \mathbf{j}_D is the canonical elementary embedding of \mathbf{V} into $\mathbf{V}^{\mathcal{H}(\chi)}/D$
- (b) \mathbf{j}_1 is the Mostowski Collapse mapping of $\mathbf{V}^{\mathcal{H}(\chi)}/D$ onto some transitive class \mathbf{M} .

Claim 0.8. *Let λ be a supercompact cardinal.*

0) Without loss of generality there is a Laver diamond $h : \lambda \rightarrow \mathcal{H}(\lambda)$, that is this holds after some λ -c.c. forcing of cardinality λ .

- 1) For some λ -c.c. forcing \mathbb{R} of cardinality λ , in $\mathbf{V}_1 = \mathbf{V}^{\mathbb{R}}$, λ is supercompact Laver indestructible which means it is supercompact not only in \mathbf{V}_1 but also in $\mathbf{V}_1^{\mathbb{Q}}$ when in \mathbf{V}_1 :

- (*) $_{\mathbb{Q}}$ \mathbb{Q} is a forcing notion which is $(< \lambda)$ -directed complete (i.e. such that any directed system of $< \lambda$ forcing conditions has a common upper bound).

- 2) Similarly to part (1) but for some pregiven $\mathbf{h} : \lambda \rightarrow \lambda$ we replace (*) $_{\mathbb{Q}}$ by:

(*) $_{\mathbb{Q}, \mathbf{h}}$ \mathbb{Q} is λ -complete and if $\mathbb{Q} \in N \prec (\mathcal{H}(\chi), \in)$, $\lambda[N] := N \cap \lambda \in \lambda$, $N^{<\lambda[N]} \subseteq N$, $|\mathbb{Q} \cap N| < \mathbf{h}(N \cap \lambda)$ and $\mathbf{G} \subseteq \mathbb{Q} \cap N$ is directed and even $\lambda[N]$ -directed generic over N then \mathbf{G} has a common upper bound in \mathbb{Q} .

3) For a given Laver diamond h there is an Easton support iteration \mathbf{q} such that:

- (A) (a) $\mathbf{q} = \langle \mathbb{P}_\alpha^*, \mathbb{Q}_\beta^* : \alpha \leq \lambda, \beta < \lambda \rangle$
- (b) $\mathbb{P}_\alpha^*, \mathbb{Q}_\alpha^* \in \mathcal{H}(\lambda)$ for $\alpha < \lambda$
- (c) each \mathbb{Q}_β^* is $(< |\beta|)$ -strategically complete
- (d) $\mathbb{Q}_\beta^* = h(\beta)$ if this choice satisfies (b), (c)
- (B) in $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}_\lambda^*}$, if \mathbb{Q} is $(< \lambda)$ -strategically complete and $\chi > \lambda$ satisfies $\mathbb{Q} \in \mathcal{H}(\chi)$, then for some normal filter D on $[\mathcal{H}(\lambda)]^{<\lambda}$, the set of N satisfying the following belongs to D :
 - (a) $N \prec (\mathcal{H}(\chi), \in)$,
 - (b) $\mathbb{Q} \in N$
 - (c) N is isomorphic to $(\mathcal{H}(\chi_N), \in)$ for some $\chi_N < \lambda$ but necessarily $\chi_N > \lambda[N] = \lambda \cap N$, moreover
 - (d) $\mathbb{Q} \restriction N \cong \mathbb{Q}_{\lambda[N]}$ that is if $\mathbf{V}_1 = \mathbf{V}[\mathbf{G}_\lambda]$, $\mathbf{G}_\lambda \subseteq \mathbb{P}_\lambda^*$ is generic over \mathbf{V} , $\mathbf{G}_\alpha = \mathbf{G}_\lambda \cap \mathbb{P}_\alpha^*$ for $\alpha < \lambda$, then $\mathbf{G}_\lambda \in N$ and in $\mathbf{V}[\mathbf{G}_{\lambda(N)}]$ we have $\mathbb{Q} \restriction N = \mathbb{Q}_{\lambda[N]}[\mathbf{G}_{\lambda[N]}]$
- (C) if in (B) we can add clause (e) then λ is supercompact also in $\mathbf{V}^{\mathbb{P}}$:
 - (e) and for every $p \in \mathbb{Q} \cap N$ there is $q \in \mathbb{Q}$ above p which is (N, \mathbb{Q}) -generic.

Proof. 0), 1) By Laver [Lav78].

2), 3) Similarly. □_{0.8}

Definition 0.9. For an ideal \mathbb{I} of subsets of X , including all singletons for simplicity, we define “the four basic cardinal invariants of the ideal”:

- (a) $\text{cov}(\mathbb{I})$, the covering number is $\min\{\theta : \text{there are } A_i \in \mathbb{I} \text{ for } i < \theta \text{ whose union is } X\}$
- (b) $\text{add}(\mathbb{I})$, the additivity of \mathbb{I} is $\min\{\theta : \text{there are } A_i \in \mathbb{I} \text{ for } i < \theta \text{ whose union is not in } \mathbb{I}\}$
- (c) $\text{cf}(\mathbb{I})$, the cofinality of \mathbb{I} is $\min\{\theta : \text{there are } A_i \in \mathbb{I} \text{ for } i < \theta \text{ such that } (\forall A \in \mathbb{I})(\exists i)(A \subseteq A_i)\}$
- (d) $\text{non}(\mathbb{I})$, the uniformity of \mathbb{I} is $\min\{|Y| : Y \subseteq X \text{ but } Y \notin \mathbb{I}\}$.

Remark 0.10. We may use e.g. $\text{cov}(\text{meagre}_\lambda)$ and $\text{cov}(\text{Cohen}_\lambda)$ for the same number.

§ 1. LIKE RANDOM REAL FORCING FOR WEAKLY COMPACT λ

We consider the question:

- Question 1.1.* 1) Is there a forcing notion which is a non-trivial λ^+ -c.c., $(< \lambda)$ -strategically complete not adding a λ -Cohen sequence from ${}^\lambda 2$.
2) Moreover is λ -bounded.

Recall that for $\lambda = \aleph_0$, “random real forcing” is such a forcing notion but we do not know to generalize measure to λ with λ -completeness or so whereas for Cohen forcing and many other definable forcing notions which add a Cohen real we know.

We have wondered about this a long time, see [Sh:945] and some papers of Roslanowski-Shelah [RoSh:777], [RoSh:860], [RoSh:888], [RoSh:942] (check relevancy). Up to recently, we were sure that the answer is negative.

Surprisingly for λ weakly compact there is a positive answer, a posteriori a straightforward one.

We define \mathbb{Q}_κ by induction on the inaccessible κ . Now for κ the first inaccessible \mathbb{Q}_κ is the κ -Cohen forcing. In fact, if for κ in $S_{\text{na}} = \{\kappa : \kappa \text{ inaccessible not a limit of inaccessible cardinals}\}$, \mathbb{Q}_κ is equivalent to the κ -Cohen forcing. But if κ is a limit of inaccessibles, the conditions are such that the generic $\eta \in {}^\kappa 2$ satisfies for many inaccessibles $\partial < \kappa$, $\eta \restriction \partial$ is somewhat ∂ -Cohen, e.g. for any sequence $\langle \mathcal{I}_\partial : \partial \in S_{\text{na}} \cap \kappa \rangle$, \mathcal{I}_∂ is a dense open subset of ${}^\partial 2$, for every large enough $\partial \in S_{\text{na}}$ we have $\eta \restriction \partial \in \mathcal{I}_\partial$. At first glance this may look ridiculous: η is made more Cohen’s, but still in the end, i.e. for κ weakly compact, it has an antithetical character.

We say more particularly on \mathbb{Q}_κ in §3.

§ 1(A). Adding a $\eta \in {}^\kappa 2$.

Notation 1.2. 1) Here ∂, κ will denote strongly inaccessible cardinals.

2) For $\mathcal{T} \subseteq {}^{>\alpha} 2$ and $\eta \in {}^{>\alpha} 2$ let $\mathcal{T}^{[\eta]} = \{\nu : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu \in \mathcal{T}\}$.

3) For $\mathcal{T} \subseteq {}^{>\delta} 2$ let $\lim_\delta(\mathcal{T}) = \{\nu \in {}^\delta 2 : (\forall \alpha < \delta)(\nu \restriction \alpha \in \mathcal{T})\}$.

Definition 1.3. We define a forcing notion $\mathbb{Q}_\kappa = \mathbb{Q}_\kappa^2$ by induction on inaccessible κ :

- (A) $p \in \mathbb{Q}_\kappa$ iff there is a witness $(\varrho, S, \bar{\Lambda})$ which means:
- (a) p is a subtree of ${}^{>\kappa} 2$, i.e. a non-empty subset of ${}^{>\kappa} 2$ closed under initial segments
 - (b) (α) $S \subseteq \kappa$ is not stationary, moreover
 - (β) $\partial < \kappa \Rightarrow S \cap \partial$ is not stationary
 - (γ) every member of S is (strongly) inaccessible
 - (c) $\varrho = \text{tr}(p)$ is the trunk of p which means:
 - (α) $\varrho \in {}^{>\kappa} 2$
 - (β) $\alpha \leq \ell g(\varrho) \Rightarrow p \cap {}^\alpha 2 = \{\varrho \restriction \alpha\}$ hence $\text{tr}(p) \in p$
 - (γ) $\varrho \restriction \langle 0 \rangle, \varrho \restriction \langle 1 \rangle$ belongs to p
 - (d) if $\varrho \leq \eta \in p$ then $\eta \restriction \langle 0 \rangle, \eta \restriction \langle 1 \rangle \in p$
 - (e) [continuity] if $\delta \in \kappa \setminus S$ is a limit ordinal $> \ell g(\varrho)$ and $\eta \in {}^{>\delta} 2$ then $\eta \in p$ iff $(\forall \alpha < \delta)(\eta \restriction \alpha \in p)$

- (f) (α) $\bar{\Lambda} = \langle \Lambda_\partial : \partial \in S \rangle$
 (β) Λ_∂ is a set of $\leq \partial$ dense open subsets of \mathbb{Q}_∂
 (g) if $\partial \in S, \partial > \ell g(q)$ then
 (α) $p \cap^{\partial > 2} \in \mathbb{Q}_\partial$
 (β) $\eta \in p \cap^{\partial > 2}$ iff $(\forall \alpha < \partial)(\eta \restriction \alpha \in p)$ and $(\forall \mathcal{I} \in \Lambda_\partial)(\exists q \in \mathcal{I})[\eta \in \lim_\partial(q)]$
 (B) $\mathbb{Q}_\kappa \models "p \leq q"$ iff $p \supseteq q$
 (C) (a) let $S[p] = \{\delta < \kappa : \delta > \ell g(\text{tr}(p)) \text{ and } \neg(\forall \eta \in {}^\delta 2)[\eta \in p \leftrightarrow (\forall \alpha < \delta)(\eta \restriction \alpha \in p)]\}$, so $S[p] \subseteq S$ when $(\text{tr}(p), S, \bar{\Lambda})$ is a witness
 (b) we say $(\text{tr}(p), S, \bar{\Lambda}, E)$ is a full witness for $p \in \mathbb{Q}_\kappa$ if $(\text{tr}(p), S, \bar{\Lambda})$ is a witness for $p \in \mathbb{Q}_\kappa$ and E is a club of κ disjoint to S and to $[0, \ell g(\text{tr}(p))]$
 (D) let \mathbb{Q}'_κ be the set of p satisfying the following: as in 1.3(A):
 there is $(\varrho, S, \bar{\Lambda})$ such that clauses (a) – (f), (g)(α) from 1.3(A) hold and
 (g)(β)' $\eta \in p \cap^{\partial > 2}$ iff $(\forall \alpha < \partial)(\eta \restriction \alpha \in p)$ and $(\forall \mathcal{I} \in \Lambda_\partial)(\exists q)[(p \cap^{\partial > 2} \leq_{\mathbb{Q}_\partial} q \in \mathcal{I})]$.

Claim 1.4. 1) For any κ and $\eta \in {}^{\kappa > 2}$ we have $({}^{\kappa > 2})^{[\eta]}$ is a member of \mathbb{Q}_κ with $\text{tr}(({}^{\kappa > 2})^{[\eta]}) = \eta$.

2) If $p \in \mathbb{Q}_\kappa$ and $\ell g(\text{tr}(p)) < \partial < \kappa$ then $p \cap^{\partial > 2}$ belongs to \mathbb{Q}_∂ .

3) If $p \in \mathbb{Q}_\kappa$ and $\eta \in p$ then $p^{[\eta]} \in \mathbb{Q}_\kappa$ and $p \leq p^{[\eta]}$ and $\text{tr}(p^{[\eta]})$ is η if $\ell g(\eta) \geq \ell g(\text{tr}(p))$ and is $\text{tr}(p)$ otherwise.

4) ${}^{\kappa > 2}$ is the minimal member of \mathbb{Q}_κ .

5) If $(\text{tr}(p), S, \bar{\Lambda})$ is a witness for $p \in \mathbb{Q}_\kappa$ and $\ell g(\text{tr}(p)) \geq \sup(S)$ then $p = ({}^{\kappa > 2})^{[\text{tr}(p)]}$.

6) Any triple $(\varrho, S, \bar{\Lambda})$ is a witness for at most one p .

7) If $(\varrho, S, \bar{\Lambda})$ satisfies clauses (c)(α), (b)(α), (β), (γ), (f)(α), (β) of Definition 1.3(A) then there is a unique $p \in \mathbb{Q}_\kappa$ which it witnesses.

8) If $(\varrho, S, \bar{\Lambda})$ witnesses $p \in \mathbb{Q}_\kappa$ then also $(\varrho, S[p], \bar{\Lambda} \restriction S[p])$ witnesses it recalling Definition 1.3(C)(a).

9) For every $p \in \mathbb{Q}_\kappa$ there is a maximal antichain \mathcal{I} to which p belongs and $q_1 \neq q_2 \in \mathcal{I} \Rightarrow \lim_\kappa(q_1) \cap \lim_\kappa(q_2) = \emptyset$ hence $\{q \in \mathbb{Q}_\kappa : p \leq_{\mathbb{Q}_\kappa} q \text{ or } \lim_\kappa(q) \cap \lim_\kappa(p) = \emptyset\}$ is dense open.

Proof. 1) Let $S = \emptyset$ so $(\eta, \emptyset, < >)$ is a witness.

2) If $(\text{tr}(p), S, \langle \Lambda_\theta : \theta \in S \rangle)$ witness $p \in \mathbb{Q}_\kappa$ then $(\text{tr}(p), S \cap \partial, \langle \Lambda_\theta : \theta \in S \cap \partial \rangle)$ witnesses $p \cap^{\partial > 2} \in \mathbb{Q}_\partial$.

3) – 8) Easy, too.

9) Let $\mathcal{I} = \{({}^{\kappa > 2})^{[\rho]} : \rho \in {}^{\kappa > 2} \setminus p \text{ and } \alpha < \ell g(\rho) \Rightarrow \rho \restriction \alpha \in p\} \cup \{p\}$. □_{1.4}

Claim 1.5. If $p_i \in \mathbb{Q}_\kappa$ and $\text{tr}(p_i) \leq \eta \in p_i$ for $i < i(*)$ and $i(*) < \ell g(\eta)$ or just there is no $\partial \leq i(*)$ which is $> \ell g(\eta)$ then $p = \cap \{p_i : i < i(*)\}$ is the $\leq_{\mathbb{Q}_\kappa}$ -lub of $\{p_i : i < i(*)\}$.

Proof. Straightforward. □

Claim 1.6. 1) If $p \in \mathbb{Q}_\kappa$ and $\rho \in p$ then there is η such that $\rho \leq \eta \in \lim_\kappa(p)$.

2) If $\bar{p} = \langle p_i : i < \delta \rangle$ is a sequence of members of \mathbb{Q}_κ , $\langle \text{tr}(p_i) : i < \delta \rangle$ is \leq -increasing and $\alpha < \delta \Rightarrow \min(S[p_\alpha] \setminus \sup\{\ell g(\text{tr}(p_i)) + 1 : i < \delta\}) > \delta$ then $p_\delta = \cap \{p_i : i < \delta\}$ is a $\leq_{\mathbb{Q}_\kappa}$ -lub of \bar{p} .

- 3) If $p \in \mathbb{Q}_\kappa$ and \mathcal{I}_i is a dense subset of \mathbb{Q}_κ for $i < i(*)$ and $i(*) < \kappa^+$ and $\rho \in p$ then there is η such that $\rho \triangleleft \eta \in \lim_\kappa(p)$ and $(\forall i < i(*))(\exists q \in \mathcal{I}_i)(\eta \in \lim_\kappa(q))$.
- 4) If $\delta < \kappa$, $p_i \in \mathbb{Q}_\kappa$ is $\leq_{\mathbb{Q}_\kappa}$ -increasing with $i < \delta$, $(\eta_i, S_i, \bar{\Lambda}_i, E_i)$ is a full witness for p_i satisfying $i < j < \delta \Rightarrow E_j \subseteq E_i \wedge \min(E_i) < \ell g(\text{tr}(p_j))$ then the sequence $\langle p_i : i < \delta \rangle$ has a $\leq_{\mathbb{Q}_\kappa}$ -upper bound.

Proof. We prove by induction on κ that the four parts of the claim hold.

1) Let $(\text{tr}(p), S, \bar{\Lambda})$ be a witness for p .

Case 1: In S there is a last member ∂ and $\partial > \ell g(\text{tr}(p))$.

If $\ell g(\rho) < \partial$ then by the induction hypothesis for ∂ there is ϱ such that $\rho \triangleleft \varrho \in p \cap {}^\partial 2$; if $\ell g(\rho) \geq \partial$ let $\varrho = \rho$. So $q = p^{[\varrho]}$ belong to \mathbb{Q}_κ by 1.4(3) and we can check by cases that $\text{tr}(q) = \varrho$. Now $q = (\kappa^{>2})^{[\varrho]}$ by 1.4(5), so the rest should be clear.

Case 2: $\sup(S) \leq \ell g(\text{tr}(p))$, e.g. S is empty.

Similarly.

Case 3: Neither Case 1 nor Case 2, i.e. $\sup(S) > \ell g(\text{tr}(p))$ and S has no last element.

Let $\theta = \text{cf}(\text{otp}(S))$ and let $\langle \alpha_\varepsilon : \varepsilon < \theta \rangle$ be increasing continuous with limit $\sup(S)$. Without loss of generality $\alpha_0 = \ell g(\text{tr}(p))$ and $\varepsilon < \theta \Rightarrow \alpha_{\varepsilon+1} \in S$ and $\omega\varepsilon < \theta \Rightarrow \alpha_{\omega\varepsilon} \notin S$; recalling that every member of S is strongly inaccessible and S is nowhere stationary this is clear. Let $\alpha_\theta = \sup(S)$ and now we choose $\eta_\varepsilon \in p \cap {}^{\alpha_\varepsilon} 2$ by induction on $\varepsilon \leq \theta$ such that $\varepsilon = 0 \Rightarrow \eta_\varepsilon = \ell g(\text{tr}(p))$ and $\zeta < \varepsilon \Rightarrow \eta_\zeta \leq \eta_\varepsilon$. For $\varepsilon = 0$ this is obvious, for ε limit $< \theta$ let $\eta_\varepsilon = \bigcup \{\eta_\zeta : \zeta < \varepsilon\}$, it belongs to p by clause (A)(e) of Definition 1.3 because $\alpha_\varepsilon \notin S$.

Lastly, for $\varepsilon = \zeta + 1$ use the induction hypothesis for part (2) for $\partial = \alpha_\varepsilon$. Having carried the induction, if $\theta = \kappa$, i.e. $\sup(S) = \kappa$ then $\eta := \bigcup \{\eta_\varepsilon : \varepsilon < \kappa\}$ is as required, and if $\theta < \kappa$, i.e. $\sup(S) < \theta$ then $\eta = \bigcup \{\eta_\varepsilon : \varepsilon < \theta\}$ belongs to ${}^{\sup(S)} 2$ and as above it belongs to p . So again by 1.4(5) we have $p^{[\eta_\theta]} = (\kappa^{>2})^{[\eta_\theta]}$ and we can easily finish.

2) Let $(\eta_i, S_i, \bar{\Lambda}_i)$ be a witness for $p_i \in \mathbb{Q}_\kappa$ for $i < \delta$, without loss of generality $S_i = S[p_i]$, see clause (C) of Definition 1.3. As \bar{p} is $\leq_{\mathbb{Q}_\kappa}$ -increasing clearly $\langle \eta_i : i < \delta \rangle$ is \leq -increasing and let $\eta_\delta = \bigcup \{\eta_i : i < \delta\}$. So $i < j < \delta \Rightarrow p_i \leq_{\mathbb{Q}_\kappa} p_j \Rightarrow \eta_i = \text{tr}(p_i) \in p_j$ hence $i < \delta \Rightarrow \eta_i = \bigcap \{p_j : j \in (i, \delta)\} = p_\delta$, hence $i < \delta \Rightarrow \eta_\delta = \bigcup \{\eta_j : j < \delta\} \in p_i$ recalling $i < \delta \Rightarrow \sup(S_i - \sup\{\ell g(\text{tr}(p_i)) + 1 : i < \delta\}) > \delta$.

Let $S := \bigcup \{S_i : i < \delta\} \setminus (\ell g(\eta_\delta) + 1)$ and $\bar{\Lambda}_i = \langle \Lambda_{i,\partial} : \partial \in S_i \rangle$ and for $\partial \in S$ let $\Lambda_\partial := \bigcup \{\Lambda_{i,\partial} : i < \delta \text{ and } \partial \in S_i\}$. So clearly Λ_∂ is a set of $\leq |\delta| \cdot \partial$ dense subsets of \mathbb{Q}_δ . Also $\partial \in S \Rightarrow \partial > \delta$ because if $\partial \in S$ then for some $i < \delta$, $\partial \in S_i$ and by an assumption $\min(S_i \setminus \sup\{\ell g(\text{tr}(p_i)) + 1 : i < \delta\}) > \delta$ hence $\partial > \delta$. Moreover, $\partial > \ell g(\eta_\delta)$. Why? If $\delta = \ell g(\eta_\delta)$ then by the previous sentence, and if $\delta < \ell g(\eta_\delta)$ then $\ell g(\eta_\delta)$ is a singular ordinal whereas S is a set of inaccessible cardinals, so $\partial \neq \ell g(\eta_\delta)$, but $\partial \geq \ell g(\eta_\delta)$ by the choice of S so indeed $\partial > \ell g(\eta_\delta)$. Together the last paragraph shows that $\eta_\delta, S, \langle \Lambda_\partial : \partial \in S \rangle$ witness that $p = \bigcap \{p_i : i < \delta\}$ belongs to \mathbb{Q}_κ ; being a $\leq_{\mathbb{Q}_\kappa}$ -lub of \bar{p} is obvious by the definition of $\leq_{\mathbb{Q}_\kappa}$.

3) Without loss of generality $i(*) = \kappa$ and let $(\text{tr}(p), S, \bar{\Lambda})$ be a witness for $p \in \mathbb{Q}_\kappa$.

First, if S is a bounded subset of κ then by part (1) which, for κ , was already proved there is $\eta \in p$ such that $\ell g(\eta) > \sup(S)$, $\ell g(\text{tr}(p))$, hence $p \leq_{\mathbb{Q}_\kappa} p^{[\eta]} = (\kappa^{>2})^{[\eta]}$; so the claim becomes a case the Baire category theorem for $\kappa^{>2}$.

Second, if $\sup(S) = \kappa$ let $\langle \alpha_\varepsilon : \varepsilon < \kappa \rangle$ be as in Case 3 of the proof of part (1), and we choose a pair (p_i, ε_i) by induction on $i < \kappa$ such that:

- (*) (a) $p_i \in \mathbb{Q}_\kappa$ and $\ell g(\text{tr}(p_i)) \leq \alpha_{\varepsilon_i}$
- (b) $j < i \Rightarrow p_j \leq_{\mathbb{Q}_\kappa} p_i$
- (c) $p_0 = p, \varepsilon_0 = 0$
- (d) if $i = 2j + 1$ then $p_i \in \mathcal{S}_j$
- (e) if $i = 2j + 2$ then $\ell g(\text{tr}(p_i)) = \alpha_{\varepsilon_i}$.

Can we carry the induction? Clearly it suffices to find p_i as required. For $i = 0$ use clause (c), for i limit use part (2) which we have already proved and let $\varepsilon_i = \sup\{\varepsilon_j : j < i\}$. For $i = 2j + 2$ recall $\ell g(\text{tr}(p_{2j+1})) \leq \alpha_{\varepsilon_{2j+1}}$; let $\varepsilon_i = \varepsilon_{2j+1} + 1$ and so by part (1) which we have already proved there exists $\eta \in p_{2j+1}$ of length α_{ε_i} and let $p_i = (p_{2j+1})^{[\eta]}$ so $p_{2j+1} \leq_{\mathbb{Q}_\kappa} p_i$ by 1.4(3).

Lastly, if $i = 2j + 1$ use $\mathcal{S}_j \subseteq \mathbb{Q}_\kappa$ is a dense subset of κ .

Having carried the induction, $\eta := \cup\{\text{tr}(p_i) : i < \delta\}$ is as required.

4) Like part (2) just easier. □_{1.6}

Claim 1.7. 1) \mathbb{Q}_κ is κ -strategically closed.

2) \mathbb{Q}_κ satisfies the κ^+ -c.c.

Proof. 1) Immediate by 1.6(4).

2) Clearly

- (*)₁ $\kappa^{>2}$ has cardinality κ (recall that κ is inaccessible)
- (*)₂ if $p_1, p_2 \in \mathbb{Q}_\kappa$ has the same trunk then they are compatible.

Together we are clearly done. □_{1.7}

Remark 1.8. Moreover $(A) \Rightarrow (B)$ where

- (A) (a) $\alpha \leq \beta < \kappa$
- (b) $\eta \in {}^\beta 2$
- (c) $p_i \in \mathbb{Q}_\kappa$ for $i < \alpha$
- (d) $(\text{tr}(p_i), S_i, \bar{\Lambda}_i)$ witness $p_i \in \mathbb{Q}_\kappa$
- (e) $\text{tr}(p_i) \leq \eta \in p_i$
- (f) $S = \cup\{S_i : i < \alpha\} \setminus \ell g(\eta)$
- (g) $\Lambda_\partial := \cup\{\Lambda_{i,\partial} : \partial \text{ satisfies } \partial \in S_i\}$ is a set of $\leq \partial$ dense subsets of \mathbb{Q}_∂
- (B) $\cap\{p_i^{[\eta]} : i < \alpha\} \in \mathbb{Q}_\kappa$ is a $\leq_{\mathbb{Q}_\kappa}$ -lub of $\{p_i : i < \alpha\}$ and has the witness $(\eta, S, \langle \Lambda_\partial : \partial \in S \rangle)$.

Claim 1.9. If κ is weakly compact then \mathbb{Q}_κ is κ -bounded, i.e. for every $f \in (\kappa^\kappa)^{\mathbf{V}[\mathbb{Q}_\kappa]}$ there is $g \in (\kappa^\kappa)^{\mathbf{V}}$ such that $f \leq g$, that is, $\alpha < \kappa \Rightarrow f(\alpha) \leq g(\alpha)$.

Proof. Let $p \Vdash "f \in {}^\kappa \kappa"$. By induction on i , we choose $p_i, \beta_i = \beta(i), \varrho_i, S_i, \bar{\Lambda}_i, E_i$ such that:

- ⊗ (a) $p_i \in \mathbb{Q}_\kappa$
- (b) $\langle \beta_j : j \leq i \rangle$ is an increasing continuous sequence of ordinals $< \kappa$
- (c) $p_0 = p$ and $\beta_0 = \ell g(\text{tr}(p)) + 1$
- (d) $(\varrho_i, S_i, \bar{\Lambda}_i, E_i)$ is a full witness for $p_i \in \mathbb{Q}_\kappa$

- (e) if $j < i$ then
 - (α) $p_j \leq_{\mathbb{Q}_i} p_i$
 - (β) $p_j \cap^{\beta(j) \geq 2} = p_i \cap^{\beta(j) \geq 2}$ hence $\varrho_i = \varrho_0$
 - (γ) $\beta_i \in E_j$
 - (δ) $E_i \subseteq E_j$
- (f) if $i = j + 1$ then $\{\alpha < \kappa : p_i \Vdash \text{"}\underline{f}(j) \neq \alpha\text{"}\}$ has cardinality $< \kappa$.

For $i = 0$ choose a full witness $(\varrho_0, S_0, \bar{\Lambda}_0, E_0)$ for p , and use clause (c), for $i = \text{limit}$ work as in the proof of 1.6(3).

For $i = j + 1$, i.e. successor we shall use the definition of “ κ is weakly compact” . Let $\langle q_{j,\beta} : \beta < \beta(*) \rangle$ be a maximal antichain (or just list a predense subset) of \mathbb{Q}_κ such that $q_{i,\beta} \Vdash \text{"}\underline{f}(j) = \gamma\text{"}$ for some $\gamma = \gamma_{i,\beta}$ and $q_{i,\beta}$ is $\leq_{\mathbb{Q}_\kappa}$ -above p_j or $\lim_\kappa(q_{i,\beta}) \cap \lim_\kappa(p) = \emptyset$, recalling 1.4(9).

But \mathbb{Q}_κ satisfies the κ^+ -c.c., see 1.7(2), so without loss of generality $\beta(*) \leq \kappa$, so without loss of generality $\beta(*) = \kappa$. By weak compactness there is a strongly inaccessible $\partial_j = \partial(j) > \beta_j$ such that $\{q_{j,\beta} \cap^{\partial(j) > 2} : \beta < \partial_j\}$ is a pre-dense subset of \mathbb{Q}_{∂_j} and $\partial_j \notin \cup\{S[q_{j,\beta}] : \beta < \partial_j\}$.

Let $\mathcal{J} = \{q \in \mathbb{Q}_{\partial_j} : \text{for some } \beta < \partial_j \text{ we have } (q_{j,\beta} \cap^{\partial(j) > 2}) \leq_{\mathbb{Q}_{\partial(j)}} q\}$, it is a dense open subset of $\mathbb{Q}_{\partial(j)}$. Let $\mathcal{X} = \{\eta : \eta \in p \cap^{\partial(j) > 2} \text{ and } (\exists \beta < \partial_j)(\eta \in \lim_{\partial_j}(q_{j,\beta} \cap^{\partial(j) > 2}))\}$.

Now for each $\rho \in \mathcal{X}$ there is $r_{j,\rho} \in \mathbb{Q}_\kappa$ with $\text{tr}(r_{j,\rho}) = \rho$ and $r_{j,\rho}$ forces a value to $\underline{f}(j)$. Indeed, there is $\beta < \partial_j$ such that $\eta \in \lim_{\partial_j}(q_{j,\beta} \cap^{\partial(j) > 2})$ so by our assumptions on the $q_{j,\beta}$ ’s necessarily $p_j \leq q_{j,\beta}$, so $q_{j,\beta}^{[\rho]}$ can serve as $r_{j,\rho}$ and is $\geq p_j$. Let $(\rho, S_{j,\rho}, \bar{\Lambda}_{j,\rho})$ witness $r_{j,\rho} \in \mathbb{Q}_\kappa$.

Lastly, let

- (a) $p_i = \cup\{r_{j,\rho} : \rho \in \mathcal{X}\}$
 - (b) $\beta_i = \partial_j + 1$
 - (c) $S_i = S'_i \cup S''_i$ where
 - $S'_i = \cup\{S[r_{j,\rho}] : \rho \in (p_i \cap^{\partial(j) > 2}) \setminus (\partial_j + 1)\}$
 - $S''_i = S_j \cap \partial_j$
 - (d) $\bar{\Lambda}_i = \langle \Lambda_{i,\partial} : \partial \in S_i \rangle$ where
 - $\Lambda_{i,\partial}$ is $\Lambda_{j,i}$ if $\partial \in S''_i$ and
 - $\Lambda_{i,\partial}$ is $\cup\{\Lambda_{j,\rho} : \rho \in p_i \cap^{\partial(j) > 2}\}$ if $\partial \in S'_i$
 - (e) E_i is a club of κ which is $\subseteq E_j \setminus \beta_i$ and is disjoint to $S[r_{j,\rho}]$ for every $\rho \in \mathcal{X}$.
- Now check.

□_{1.9}

§ 1(B). Adding a dominating member of $\prod_{\varepsilon < \kappa} \theta_\varepsilon$.

Here we present a variant of the forcing from §(1A), this time dealing with sequences from $\prod_{\varepsilon < \lambda} \theta_\varepsilon$ instead of ${}^\lambda 2$ and we have an $|\varepsilon|^+$ -complete filter D_ε on θ_ε for $\varepsilon < \lambda$.

Note that Definitions 1.12, 1.13 are used in §2, too. Also note that $\mathbb{Q}_{\bar{\theta}}$ is the “one step” forcing on which we shall build later.

Remark 1.10 (Here?). For $\bar{\theta} = \langle \theta_\alpha : \alpha < \kappa \rangle$, $\mathbb{Q}_{\bar{\theta}}$ was designed to make the old κ -reals κ -meagre, we still have to expect it to behave like randoms and do this indeed.

Hypothesis 1.11. We have (for this section) a fixed 1-ip (iteration parameter) where

Definition 1.12. We say \mathfrak{r} is a 1-ip when \mathfrak{r} consists of:

- (A) λ , a weakly compact cardinal
- (B) $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ where $\varepsilon < \lambda \Rightarrow (2 \leq \theta_\varepsilon < \aleph_0) \vee (\varepsilon < \theta_\varepsilon = \text{cf}(\theta_\varepsilon) < \lambda)$
- (C) $S_* = S_{\mathfrak{r}} \subseteq \lambda$ a stationary set of strongly inaccessible cardinals satisfying $\zeta < \kappa \in S \Rightarrow \prod_{\varepsilon < \zeta} \theta_\varepsilon < \kappa$
- (D) (a) $\diamond_{S_*, I_\lambda^{\text{wc}}}$, i.e. diamond on S_* holds even modulo the weakly compact ideal, see 1.13(1) below or just
 (b) $\bar{\mathcal{P}} = \langle \mathcal{P}_\kappa \subseteq \mathcal{P}(\mathcal{H}(\kappa)) : \kappa \in S_* \rangle$ is I_λ^{wc} -positive, see Definition 1.13(3) below, so necessarily $S_* \in (I_\lambda^{\text{wc}})^+$; the default value is $\mathcal{P}_\kappa = \mathcal{P}(\mathcal{H}(\kappa))$
- (E) $S_{\mathfrak{r}}^* := \{\kappa \leq \lambda : \kappa \text{ weakly compact and } S_* \cap \kappa \in (I_\kappa^{\text{wc}})^+ \text{ moreover } \bar{\mathcal{P}} \restriction (S_* \cap \kappa) \text{ is as in clause (D)(b)}\}$.

If $\kappa \in S_{\mathfrak{r}}^*$ we may say “ κ is \mathfrak{r} -weakly compact”.

Definition 1.13. 1) Recall the weakly compact ideal on λ is $I_\lambda^{\text{wc}} = \{A \subseteq \lambda : \text{for some first order formula } \varphi(X, Y) \text{ and } A \subseteq \mathcal{H}(\lambda) \text{ we have } (\forall X \subseteq \mathcal{H}(\lambda))(\mathcal{H}(\lambda) \models \varphi(X, A)) \text{ but for no strongly inaccessible } \kappa \in A \text{ do we have } (\forall X \subseteq \mathcal{H}(\kappa))(\mathcal{H}(\kappa) \models \varphi(X, A \cap \mathcal{H}(\kappa)))\}$.

2) So 1.12(D)(a) means that some $\bar{A} = \langle A_\alpha : \alpha \in S_* \rangle$ is an I_λ^{wc} -diamond sequence, which means: for every $A \subseteq \mathcal{H}(\kappa)$ the set $\{\kappa \in S_* : A \cap \mathcal{H}(\kappa) = A_\kappa\}$ is $\neq \emptyset$ mod I_λ^{wc} .

3) We say $\bar{\mathcal{P}} = \langle \mathcal{P}_\alpha : \alpha \in S_* \rangle$ is I_λ^{wc} -positive when $S_* \in (I_\lambda^{\text{wc}})^+$ and moreover if $\varphi(X, Y)$ first order, $A \subseteq \mathcal{H}(\lambda)$ satisfies $X \subseteq \mathcal{H}(\lambda) \Rightarrow (\mathcal{H}(\lambda), \in) \models \varphi(X, A)$ then $(\exists I_\lambda^{\text{wc}} \kappa \in S_*)[A \cap \mathcal{H}(\kappa) \in \mathcal{P}_\kappa \text{ and } X \subseteq \mathcal{H}(\kappa) \Rightarrow (\mathcal{H}(\kappa), \in) \models \varphi[X, A \cap \mathcal{H}(\kappa)]]$.

4) Let $\mathbf{T}_\alpha = \prod_{\varepsilon < \alpha} \theta_\varepsilon$ for $\alpha < \kappa$ and $\mathbf{T}_{<\alpha} = \cup \{\mathbf{T}_\beta : \beta < \alpha\}$ for $\alpha \leq \kappa$, recall that for this sub-section we fix up $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$, see clause (B) of 1.12.

Convention 1.14. 1) Let κ, ∂ denote strongly inaccessible cardinals $\leq \lambda$.

2) Always p is a subtree of \mathbf{T}_κ , for some $\kappa \leq \lambda$, equivalently it belongs to $\in \mathbb{Q}_\kappa$ for some $\kappa \leq \lambda$ and for $\eta \in p$ let $p^{[\eta]} = \{\nu \in p : \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$.

Definition 1.15. We define the forcing notion $\mathbb{Q}_\kappa = \mathbb{Q}_\kappa^1$ by induction on $\kappa \leq \lambda$ as follows:

- (A) $p \in \mathbb{Q}_\kappa$ iff some $S \subseteq \kappa \cap S_*$ witnesses it, which means
 - (a) p is a subtree of $\mathbf{T}_{<\kappa}$
 - (b) p has trunk $\text{tr}(p) \in \mathbf{T}_{<\kappa}$ that is
 - $\beta \leq \ell g(\text{tr}(p)) \Rightarrow p \cap \mathbf{T}_\beta = \{\text{tr}(p) \restriction \beta\}$ but
 - $(\exists \geq^2 \alpha)(\text{tr}(p) \restriction \alpha \in p)$

- (c) if $\eta \in p \wedge \ell g(\text{tr}(p)) \leq \ell g(\eta) < \beta < \kappa$ then $(\exists \nu)(\eta \triangleleft \nu \in p \cap \mathbf{T}_\beta)$, follows from the rest
- (d) if $\eta \in p$ and $\ell g(\text{tr}(p)) < \ell g(\eta) < \kappa$ then
 - if $\theta_{\ell g(\eta)} \geq \aleph_0$ then $(\forall^\infty i < \theta_{\ell g(\eta)})(\eta \hat{\ } \langle i \rangle \in p)$
 - if $\theta_{\ell g(\eta)} < \aleph_0$ then $(\forall i < \theta_{\ell g(\eta)})(\eta \hat{\ } \langle i \rangle \in p)$
- (e) if $\delta \in \kappa \setminus S$ is a limit ordinal and $\eta \in \mathbf{T}_\delta := \prod_{\varepsilon < \delta} \theta_\varepsilon$ then $\eta \in p \Leftrightarrow (\forall \beta < \delta)(\eta \upharpoonright \beta \in p)$
- (f) if $\partial \in \kappa \cap S$ hence $\partial \in S_*$ so is strongly inaccessible, then $p \cap \mathbf{T}_\partial \in \mathbb{Q}_\partial$ and for some predense subsets \mathcal{J}_i of \mathbb{Q}_∂ for $i < i_* \leq \partial$, [if we have \mathcal{P} also $\mathcal{J} \in \mathcal{P}_\kappa$ see below] for every $\eta \in \mathbf{T}_\partial$ we have:
 - $\eta \in p$ iff $(\forall \beta < \partial)(\eta \upharpoonright \beta \in p)$ and $(\forall i < i_*)(\exists q \in \mathcal{J}_i)(\forall \beta < \partial)(\eta \upharpoonright \beta \in q)$
- (g) $S \subseteq \kappa \cap S_*$ is not stationary in any $\partial \leq \kappa$, yes also for $\partial = \kappa$, equivalently for any limit $\delta \leq \kappa$ as S_* is a set of inaccessibles and $S \subseteq S_*$
- (B) $\leq_{\mathbb{Q}_\kappa}$ is inverse inclusion
- (C) (a) let \mathcal{J} be a subset of κ . \mathcal{J} has the narrow extension property if for every $\nu \in \mathbf{T}_{<\kappa}$ there exists at most one condition $p \in \mathcal{J}$ with $\nu = \text{tr}(p)$, such that: $\{(\nu,]\text{eta}) : \text{tr}(p) = \nu \wedge \exists p \in \mathcal{J} \text{ so that } \eta \in p\}$
- (b) for $\kappa \in S_*$ we define: $\mathcal{P}'_\kappa = \{\mathcal{J} : \mathcal{J} \subseteq \mathbb{Q}_\kappa, \mathcal{J} \text{ is dense and has the narrow extension property}\}$.

Claim 1.16. 1) $\mathbf{T}_{<\kappa}$ belongs to \mathbb{Q}_κ and $p \in \mathbb{Q}_\kappa \Rightarrow \mathbb{Q}_\kappa \models \text{“}\mathbf{T}_{<\kappa} \leq p\text{”}$ and $\eta \in p \in \mathbb{Q}_\kappa \Rightarrow p \leq_{\mathbb{Q}_\kappa} p^{[\eta]} \in \mathbb{Q}_\kappa$.

2) For $p \in \mathbb{Q}_\kappa$ and $\alpha < \kappa$ the set $\{p^{[\eta]} : \eta \in p \cap \mathbf{T}_\alpha\}$ is predense in \mathbb{Q}_κ above p .

2A) If $p \in \mathbb{Q}_\kappa$ and $\ell g(\text{tr}(p)) < \partial < \kappa$ then $p \cap \mathbf{T}_\partial \in \mathbb{Q}_\partial$.

3) \mathbb{Q}_κ is a forcing notion, satisfies the κ^+ -c.c. moreover if $p, q \in \mathbb{Q}_\kappa$ have the same trunk then p, q are compatible, in fact, $p \cap q$ belongs to \mathbb{Q}_κ and is a $\leq_{\mathbb{Q}_\kappa}$ -lub with the same trunk.

4) If $\nu \in \mathbf{T}_\gamma$ and $p_i \in \mathbb{Q}_\kappa, \text{tr}(p_i) = \nu$ for $i < i(*)$ then $p = \cap \{p_i : i < i(*)\}$ belongs to \mathbb{Q}_κ has the trunk ν and is a $\leq_{\mathbb{Q}_\kappa}$ -lub of $\{p_i : i < i(*)\}$ provided that at least one of the following holds:

- $i(*) \leq \gamma$
- $(\forall \varepsilon)[\ell g(\nu) \leq \varepsilon < \kappa \wedge \theta_\varepsilon > \aleph_0 \rightarrow i(*) < \theta_\varepsilon]$ and $(i(*) < \min(S_* \setminus (\ell g(\nu) + 1)))$.

4A) $p, q \in \mathbb{Q}_\kappa$ are incompatible iff $\text{tr}(p) \notin q \vee \text{tr}(q) \notin p$.

4B) If $\nu \in \mathbf{T}_\gamma, p_i \in \mathbf{T}_\gamma$ and $\text{tr}(p_i) \triangleleft \nu \in p_i$ then $p = \cap \{p_i^{[\nu]} : i < \delta(*)\}$ is a common lub of $\{p_i : i < i(*)\}$ in \mathbb{Q}_κ and has trunk ν .

5) $\eta = \cup \{\text{tr}(p) : p \in \mathbf{G}_{\mathbb{Q}_\kappa}\}$ is a \mathbb{Q}_κ -name of a member of $\prod_{\varepsilon < \kappa} \theta_\varepsilon$.

6) If $\nu \in \prod_{\varepsilon < \kappa} \theta_\varepsilon$ then $\Vdash_{\mathbb{Q}_\kappa}$ “for arbitrarily large $\varepsilon < \lambda$ we have $\eta(\varepsilon) \neq \nu(\varepsilon)$ and for every $\varepsilon < \lambda$ large enough $\theta_\varepsilon \geq \aleph_0 \Rightarrow \eta(\varepsilon) > \nu(\varepsilon)$ ”.

7) η is a new branch of $\mathbf{T}_{<\kappa}$ and is generic for \mathbb{Q}_κ , i.e. $\mathbf{G} = \{p \in \mathbb{Q}_\kappa : \eta \text{ is a branch of } p\}$.

8) \mathbb{Q}_κ is $(< \kappa)$ -strategically complete.

Proof. 1), 2), 2A) Straight; on parts (3),(4),(4A), see more in 1.17, 1.20, 1.21.

3) By (4B) and the number of possible trunks of $p \in \mathbb{Q}_\kappa$ is $|\mathbf{T}_{<\kappa}| = \kappa$.

4) By (4B).

4A) Clearly if $\text{tr}(p) \notin q$ then p, q are incompatible, and similarly if $q \notin \text{tr}(p)$ so the implication “if” holds. For the other direction assume $\text{tr}(p) \in q \wedge \text{tr}(q) \in p$, and we shall prove that p, q are compatible. By symmetry without loss of generality $\ell g(\text{tr}(p)) \leq \ell g(\text{tr}(q))$, let $\nu = \text{tr}(q)$, now $p^{[\nu]}, q = q^{[\nu]}$ have the same trunk, so we are done by part (3).

4B) Let S_i be a witness for $p_i \in \mathbb{Q}_\kappa$, and let $S = \cup\{S_i : i < i(*)\} \setminus (\ell g(\nu) + 1)$ and we shall prove that S witness that $p = \cap\{p_i : i < i(*)\}$ belongs to \mathbb{Q}_κ , then we are done as obviously $i < i(*) \Rightarrow p \subseteq p_i$ by the choice of p .

If $\partial \leq \ell g(\nu)$ then $\partial \cap S = \emptyset$ and if $\ell g(\nu) < \partial < \kappa$, then each $S_i \cap \partial$ is not a stationary subset of ∂ for $i < i(*)$. Also $i(*) < \partial$.

[Why? If $i(*) \leq \ell g(\nu)$ clear, if $i(*) > \ell g(\nu)$, then $S_* \cap [\ell g(\nu), i(*)] = \emptyset$ by assumption as $\partial > \ell g(\nu)$ clearly $i(*) < \partial$.] Together also $S = \cup\{S_i : i < i(*)\}$ is not stationary in ∂ ; that is, clause (g) of 1.15(A) holds.

Now obviously p is a subtree of $\mathbf{T}_{<\kappa}$, i.e. (a) of 1.15(A) holds. Also obviously $\alpha \leq \ell g(\nu) \Rightarrow p \cap \mathbf{T}_\alpha = \{\nu \restriction \alpha\}$ and $p \cap \mathbf{T}_{\ell g(\nu)+1} \subseteq \{\nu^\wedge \langle \iota \rangle : \iota < \theta_{\ell g(\nu)}\}$. If $\theta_{\ell g(\nu)} < \aleph_0$ then clearly $n < \theta_\varepsilon \wedge j < i(*) \Rightarrow \nu^\wedge \langle n \rangle \in p_i$ hence $\{\nu^\wedge \langle \iota \rangle : \iota < \theta_\varepsilon\} \subseteq p \cap \mathbf{T}_{\ell g(\nu)+1}$ so equality holds so ν is indeed the trunk of p and 1.15(A)(b) holds.

If $\theta_\varepsilon \geq \aleph_0$ then $\theta_{\ell g(\bar{\nu})} = \text{cf}(\theta_{\ell g(\nu)}) > \ell g(\nu)$ for each $i < i(*)$ there is $\iota(i) < \theta_\varepsilon$ such that $\{\nu^\wedge \langle \iota \rangle : \iota \in [\iota(i), \theta_\varepsilon]\} \subseteq p_i$ so $\iota(*) = \sup\{\iota(i) : i < i(*)\} < \theta_\varepsilon$ and $\{\nu^\wedge \langle \iota \rangle : \iota \in [\iota(*), \theta_\varepsilon]\} \subseteq p$ and again $\text{tr}(p)$ is well defined and equal to ν , so 1.15(b) holds.

The proof of clause 1.15(A)(d) is similar and clause 1.15(A)(c) follows from the rest, see part (8).

The proofs of clauses (e),(f) are straight. □_{1.17}

Claim 1.17. *For every strongly inaccessible $\kappa \leq \lambda$, the forcing notion \mathbb{Q}_κ is κ -strategically complete.*

Proof. By Claim 1.21 below. □_{1.17}

Observation 1.18. *1) If $p \leq_{\mathbb{Q}_\kappa} q$ and S is a witness for q and $\text{tr}(p) = \text{tr}(q)$ then S is a witness for p .*

Remark 1.19. We can also use \mathbb{Q}'_κ which is equivalent to \mathbb{Q}_κ and is $(< \lambda)$ -complete where we define \mathbb{Q}'_κ by:

- (A) the set of members is $\{(p, E) : p \in \mathbb{Q}_\kappa \text{ and } E \text{ is a club of } \lambda \text{ disjoint to some witness } S \text{ for } p \in \mathbb{Q}_\kappa\}$
- (B) $\mathbb{Q}'_\kappa \models “(p_1, E_1) \leq (p_2, E_2)”$ iff $p_1 \leq_{\mathbb{Q}_\kappa} p_2 \wedge E_1 \supseteq E_2$.

Definition 1.20. For strongly inaccessible $\kappa \leq \lambda$.

1) Let $\mathbf{S}_{\kappa, \gamma}^{\text{inc}}$ be the set of sequences $\langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma \rangle$ satisfying¹

- (a) $p_\alpha \in \mathbb{Q}_\kappa$
- (b) $q_\alpha \in \mathbb{Q}_\kappa$
- (c) $\beta < \alpha \Rightarrow q_\beta \leq_{\mathbb{Q}_\kappa} p_\alpha$

¹may add: (h) if $\delta < \gamma$ is a limit ordinal then $p_\delta = \cap\{p_\alpha : \alpha < \delta\}$, we do not use this

- (d) E_α is a club of κ disjoint to some witness for $q_\beta \in \mathbb{Q}_\kappa$ for every $\beta < \alpha$
 - (e) $p_\alpha \leq_{\mathbb{Q}_\kappa} q_\alpha$
 - (f) $\ell g(\text{tr}(p_\alpha)) \geq \alpha$
 - (g) $\ell g(\text{tr}(p_\alpha)) \in \cap \{E_\beta : \beta < \alpha\}$.
- 2) Let $\mathbf{S}_{\kappa, < \gamma}^{\text{inc}} = \cup \{\mathbf{S}_{\kappa, \beta} : \beta < \gamma\}$ and $\mathbf{S}_\kappa^{\text{inc}} = \mathbf{S}_{\kappa, < \kappa}^{\text{inc}}$.
- 3) For $\gamma \leq \kappa$ let $\mathbf{S}_{\kappa, \gamma}^{\text{pr}}$ be the set of sequences $\langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma \rangle$ such that
- (a) $p_\alpha, q_\alpha \in \mathbb{Q}_\kappa$ has trunk $\text{tr}(p_0)$
 - (b) E_α is a club of κ disjoint to $\ell g(\text{tr}(p_0))$ such that for every $\beta < \alpha$, E_α is disjoint to some witness of $q_\beta \in \mathbb{Q}_\kappa$
 - (c) $\min(E_\alpha) \geq \alpha$ is increasing (for transparency)
 - (d) $p_\alpha \leq_{\mathbb{Q}_\kappa} q_\alpha$
 - (e) $q_\beta \leq_{\mathbb{Q}_\kappa} p_\alpha$ when $\beta < \alpha$
 - (f) if $\beta < \alpha$ then $q_\beta \cap \mathbf{T}_{\min(E_\beta)} \subseteq p_\alpha$
 - (g) if $\delta < \gamma$ is a limit ordinal then $p_\delta = \cap \{p_\alpha : \alpha < \delta\}$ and $p_\delta \cap \mathbf{T}_{\min(\cap \{E_\alpha : \alpha < \delta\})} \subseteq q_\beta$ for $\beta \in [\alpha, \gamma)$.
- 4) $\mathbf{S}_{\kappa, < \gamma}^{\text{pr}} = \cup \{\mathbf{S}_{\kappa, \beta}^{\text{pr}} : \beta < \gamma\}$ and $\mathbf{S}_\kappa^{\text{pr}} = \cup \{\mathbf{S}_{\kappa, \gamma}^{\text{pr}} : \gamma < \kappa\}$.

Claim 1.21. 1) For every $p \in \mathbb{Q}_\kappa$ the sequence $\langle (p, p, \kappa) \rangle$ belongs to $\mathbf{S}_\kappa^{\text{inc}}$.

2) $\mathbf{S}_\kappa^{\text{inc}}$ is closed under unions of \triangleleft -increasing chains of length $< \kappa$.

3) If $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \beta \rangle \in \mathbf{S}_\kappa^{\text{inc}}$ then for some p_β and E_β we have: if $p_\beta \leq q$ and E_β is a club of κ disjoint to some witness of q or just of p_β or just of q_γ for every $\gamma < \beta$ then $\bar{\mathbf{x}}^\wedge \langle (p_\beta, q_\beta, E_\beta) \rangle \in \mathbf{S}_\kappa^{\text{inc}}$.

Proof. 1) Note that clause (d) of Definition 1.20(1) is trivially satisfied because $\gamma = 1$ here.

2) Obvious.

3) If β is a successor ordinal this is easier, so we assume β is a limit ordinal. Let $\nu_\alpha = \text{tr}(q_\alpha)$ for $\alpha < \beta$ hence $\langle \nu_\alpha : \alpha < \beta \rangle$ is a \triangleleft -increasing sequence of members of $\mathbf{T}_{< \kappa}$ and $\ell g(\nu_\alpha) \geq \alpha$. Hence $\nu_\beta := \cup \{\nu_\alpha : \alpha < \beta\} \in \mathbf{T}_{\leq \kappa}$ has length $\geq \beta$. As $\beta < \kappa$ and κ is regular, necessarily $\ell g(\nu_\beta) < \kappa$ so $\nu_\beta \in \mathbf{T}_{< \kappa}$. Also recall $\alpha_1 < \alpha_2 < \beta \Rightarrow \nu_{\alpha_2} \in E_{\alpha_1}$, but E_{α_1} is a club of κ hence $\alpha_1 < \beta \Rightarrow \ell g(\nu_\beta) \in E_{\alpha_1}$. As $\alpha_1 + 1 < \alpha_2 < \beta \Rightarrow \nu_{\alpha_2} \in q_{\alpha_1}$ and E_{α_1+1} is disjoint to a witness for q_{α_1} and by the previous sentence $\ell g(\nu_\beta) \in E_{\alpha_1+1}$ we can deduce $\nu_\beta = \cup \{\nu_{\alpha_2} : \alpha_2 \in (\alpha_1 + 1, \beta)\} \in q_{\alpha_1}$. So clearly $\nu_\beta = \bigcap_{\alpha < \delta} q_\alpha$ hence $\langle q_\alpha^{[\nu_\beta]} : \alpha < \beta \rangle$ is an increasing sequence of members of \mathbb{Q}_κ

with fixed trunk ν_β of length $\geq \beta$ as $\alpha < \beta \Rightarrow \ell g(\nu_\beta) \geq \ell g(\nu_\alpha) = \ell g(\text{tr}(q_\alpha)) \geq \alpha$, see 1.20(1)(f). So by 1.17(4) we have $p_\beta := \cap \{q_\alpha^{[\nu_\beta]} : \alpha < \beta\} \in \mathbb{Q}_\kappa$ has trunk ν_β and is equal to $\cap \{q_\alpha : \alpha < \beta\}$. Let $E_\beta = \cap \{E_\alpha : \alpha < \beta\}$ and clearly p_β, E_β are as required. Note that if $\ell g(\nu_\beta) = \beta \in S_*$ then we have the “ $i_* \leq \delta$ ” rather than “ $i_* < \delta$ ” in clause (f) of (A) of Definition 1.15. $\square_{1.21}$

Claim 1.22. 1) For every $p \in \mathbb{Q}_\kappa$ the sequence $\langle (p, p, \kappa) \rangle$ belongs to $\mathbf{S}_\kappa^{\text{pr}}$.

2) If $\gamma < \kappa$ and $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma \rangle \in \mathbf{S}_{\kappa, \gamma}^{\text{pr}}$ then there are (p_γ, E) with E a club of κ and $p_\gamma = \cap \{p_\alpha : \alpha < \gamma\}$ such that: if $p_\gamma \leq q_\gamma, \beta < \gamma \Rightarrow q_\beta \cap \mathbf{T}_{\leq \min(E_\gamma)} \subseteq q_\gamma$ and $E_\gamma \supseteq E$ a club of κ then $\bar{\mathbf{x}}^\wedge \langle (p_\gamma, q_\gamma, E_\gamma) \rangle \in \mathbf{S}_\kappa^{\text{pr}}$.

3) The union of a \triangleleft -increasing sequence of members of $\mathbf{S}_\kappa^{\text{pr}}$ of length $< \kappa$ belongs to $\mathbf{S}_\kappa^{\text{pr}}$.

3A) If $\langle \bar{x}_\beta : \beta < \delta \rangle$ is \trianglelefteq -increasing, so $\bar{x}_\beta = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma_\beta \rangle \in \mathbf{S}_\kappa^{\text{pr}}$ and $\langle \gamma_\beta : \beta < \delta \rangle$ is \leq -increasing and $\gamma := \cup\{\gamma_\beta : \beta < \delta\} < \kappa$ then $\langle (p_\alpha, q_\alpha, E_\alpha) : \alpha < \gamma \rangle \in \mathbf{S}_{\kappa, \gamma}^{\text{pr}}$.

3B) If in (3A), $\gamma = \kappa$ then $p_\kappa = \cap\{p_\alpha : \alpha < \kappa\}$ belongs to \mathbb{Q}_κ and is a $\leq_{\mathbb{Q}_\kappa}$ -lub of $\{p_\alpha, q_\alpha : \alpha < \kappa\}$.

Proof. Straight. $\square_{1.22}$

Crucial Claim 1.23. If $\kappa = \lambda$ or just $\kappa \in S_*$, $\gamma < \kappa$, $\bar{x} = \langle (p_\alpha, q_\alpha, E_\alpha) : \alpha \leq \gamma \rangle \in \mathbf{S}_{\kappa, \gamma+1}^{\text{inc}}$ and τ is a \mathbb{Q}_κ -name of a member of \mathbf{V} then we can find $(p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1})$ such that

- (a) $\bar{x} \wedge \langle (p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1}) \rangle \in \mathbf{S}_\kappa^{\text{pr}}$
- (b) if $\eta \in q_{\gamma+1} \cap \mathbf{T}_{\min(E_{\gamma+1})}$ then $q_{\gamma+1}^{[\eta]}$ forces a value to τ .

Proof. Let

$$(*)_1 \quad \mathcal{Y} = \{\text{tr}(p) : p \in \mathbb{Q}_\kappa \text{ forces a value to } \tau \text{ and } \text{tr}(p) \text{ has length } > \min(E_\gamma)\}$$

for $\eta \in \mathcal{Y}$ let p_η^* exemplify $\eta \in \mathcal{Y}$, i.e.

$$(*)_2 \quad \text{tr}(p_\eta^*) = \eta \text{ and } p_\eta^* \text{ forces a value to } \tau, \text{ necessarily } \ell g(\eta) > \min(E_\gamma).$$

Clearly

- (*)₃ (a) $\mathcal{Y} \subseteq \mathbf{T}_{<\kappa}$
- (b) if $p \in \mathbb{Q}_\kappa$ then for some $\eta \in \mathcal{Y}$ we have $\text{tr}(p) \trianglelefteq \eta \in p$.

By the Hypothesis 1.11, there is $\partial \in S_* \cap \kappa \cap E_\gamma$ but $> \min(E_\gamma)$ such that letting $\mathcal{Y}_\partial = \mathcal{Y} \cap \mathbf{T}_{<\partial}$ we have

- (*)₄ (a) $\ell g(\text{tr}(p_\gamma)) < \partial$
- (b) if $p \in \mathbb{Q}_\partial$ then $\{\eta : \text{tr}(p) \trianglelefteq \eta \in p\} \cap \mathcal{Y}_\partial \neq \emptyset$
- (c) if we have $\bar{\mathcal{P}}$ then $\{(\eta, \nu) : \eta \in \mathcal{Y} \cap \mathbf{T}_{<\partial} \text{ and } \nu \in p_\eta^* \cap \mathbf{T}_{<\partial}\} \in \mathcal{P}_\partial$.

Define:

- $q_{\gamma+1} = p_{\gamma+1}$
- $p_{\gamma+1} = \{\eta \in p_\gamma : \text{if } \ell g(\eta) \geq \partial \text{ and } \zeta < \partial \text{ is minimal such that } \eta \restriction \zeta \in \mathcal{Y} \text{ then } p_{\eta \restriction \zeta}^* \leq_{\mathbb{Q}_\kappa} p^{[\eta \restriction \zeta]}\}$
- $E_{\gamma+1} \subseteq E_\gamma \setminus (\partial + 1)$ is a club of κ such that if $\eta \in q_{\gamma+1} \cap \mathbf{T}_{<\partial}$ then $E_{\gamma+1}$ is disjoint to some witness for p_η^* .

Clearly $(p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1})$ is as required. $\square_{1.23}$

Claim 1.24. If $\kappa \in S_*$ then \mathbb{Q}_κ is $(^\lambda \lambda)$ -bounded, i.e. $\Vdash_{\mathbb{Q}_\kappa} “(\kappa \kappa)^\mathbf{V} \text{ is } \leq_{J_\kappa^{\text{bd}}} \text{-cofinal in } {}^\kappa \kappa”$.

Proof. By 1.23 and Claim 1.22. $\square_{1.24}$

Conclusion 1.25. 1) If λ is a weakly compact cardinal then there is a $(< \lambda)$ -strategically complete, λ^+ -c.c., ${}^\lambda\lambda$ -bounding forcing notion (hence not adding a λ -Cohen), and of course, adding a new $\eta \in {}^\lambda 2$.

2) In fact the forcing is λ -Borel and is λ -strategically complete hence is equivalent to a $(< \lambda)$ -complete forcing notion (which necessarily is λ^+ -c.c. $({}^\lambda\lambda)$ -bounding adding a new subset to λ). Also the forcing is definable even without parameters.

Proof. 1) Choose e.g. $\theta_\varepsilon = 2$ for $\varepsilon < \lambda$, let $S_* = \{\kappa < \lambda : \kappa \text{ is strongly inaccessible}\}$, so Hypothesis 1.11 holds.

Let $\mathbb{Q} = \mathbb{Q}_\lambda$, it is $(< \lambda)$ -strategically complete by 1.17, it is λ^+ -c.c. by 1.17(3), it is ${}^\lambda\lambda$ -bounding by 1.24, and lastly $\Vdash_{\mathbb{Q}}$ “ $\eta \in \prod_{\varepsilon < \lambda} \theta_\varepsilon$ is new” by 1.17(7).

2) See 1.19, and in general see [Sh:f, Ch.XIV].

□_{1.25}

§ 2. ADDING MANY NEW SUBSETS

We continue §1 but do not directly rely on it, in fact the forcing in §1 is not a special case of the present one but a variant of it is, when we essentially make it preserved by permutations of κ . We have two distinct though related aims. One is to have a forcing as in §1 adding some $\mu > \lambda$ subsets of λ . Second, is to solve the following problems from [Sh:945]: get the consistency of $\mathfrak{b}_\lambda > \text{cov}_\lambda(\text{meagre})$ for λ strongly inaccessible.

For the first aim here we use just “ λ is weakly compact” to prove that there is a $(< \lambda)$ -strategically complete λ^+ -c.c. for λ -bounding forcing notion making $\text{cov}_\lambda(\text{meagre})$ be at most some $\mu = \text{cf}(\mu) > \lambda$. If we start with a universe with $\mathfrak{d}_\lambda = \mathfrak{b}_\lambda > \mu$ we are done. But for the forcing increasing \mathfrak{d} (to at least λ^{++}) we need now (as the forcing is λ -bounding only when λ is weakly compact), to start with more than “ λ weakly compact”.

An additional point is that above we can get $\mathfrak{b}_\lambda > \mu = \text{cov}_\lambda(\text{meagre}) = \lambda^+$, but what if we like it to be e.g. $\mu = \lambda^{++}$? In terms of the forcing defined below, it suffices to have $\Vdash_{\mathbb{Q}_{u,\lambda}} \text{“cov}_\lambda(\text{meagre}) = |u| \text{”}$ when $|u| = \text{cf}(\text{otp}(u)) > \lambda$? So we delay this and other extensions to Part II; however using general $\bar{\theta}$ is required.

Note that here we do not have memory for 2-ip \mathfrak{r} ; a weak memory can sneak in if we use the “just” in 2.1(F), so in Definition 2.2. Note the memory seems weaker than in the second part, but the latter does not subsume, it covers cases like 3-ip \mathfrak{r} in which the order on μ is not the standard but any partial order. Still there is some memory for 3-ip \mathfrak{r} in the sense that the restriction on $\eta_\alpha \restriction \partial$ depends only on $\langle \eta_\beta \restriction \partial : \beta \in u \cap \alpha \rangle$, in §3 we consider $\langle \eta_\beta \restriction \partial_1 : \beta \in u \cap \alpha \rangle$, and possibly $\partial_1 > \partial$.

Note that below, if e.g. $u = \kappa$, the sequence $\langle \eta_\alpha(0) : \alpha < \kappa \rangle$ is not a κ -Cohen, because the $y \in \Xi_{u,\kappa}$ forbid it.

Definition 2.1. We say \mathfrak{r} is 2-ip \mathfrak{r} when:

- (A) like 1.12 but for simplicity we use the default value $\mathcal{P}_\kappa = \mathcal{P}(\mathcal{H}(\kappa))$ for $\kappa \in S_*$ and let $S_\mathfrak{r} = S_*$
- (B) $\bar{\theta} = \langle \theta_{\alpha,\varepsilon} : \alpha < \mu, \varepsilon < \lambda \rangle$ where $\theta_{\alpha,\varepsilon}$ is finite or is a regular cardinal from $[\aleph_0 + |\varepsilon|^+, \lambda)$ and let $\bar{\theta}_\alpha = \langle \theta_{\alpha,\varepsilon} : \varepsilon < \lambda \rangle$. Note that we may fix $\bar{\theta}_\alpha$ but there is no real reason to do this
- (C) $\mu > 0$ but the interesting case is $\mu > \lambda$
- (D) $\bar{D} = \langle D_{\bar{\eta}} : \bar{\eta} \in \mathbf{T}_{\subseteq \mu, < \lambda} \rangle$, see 2.6(2) below
- (E) if $\bar{\eta} \in \mathbf{T}_{u,\alpha}$ then $D_{\bar{\eta}}$ is a $(\aleph_0 + |\alpha|)^+$ -complete filter on $\text{suc}(\bar{\eta}) = \{\bar{\nu} \in \mathbf{T}_{u,\alpha+1} : \bar{\nu} \restriction \alpha = \bar{\eta}\}$, see 2.6(3) below
- (F) if $\bar{\eta} \in \mathbf{T}_{u_2,\alpha}$ and $u_1 \subseteq u_2$ and $\bar{\eta}_2 \restriction u_2 = \bar{\eta}_1$ then $D_{\bar{\eta}_1} = \{\{\bar{\nu} \restriction u_1 : \bar{\nu} \in X\} : X \in D_{\bar{\eta}_2}\}$ or just \subseteq .

Definition 2.2. 1) We say \mathfrak{r} is 3-ip when: as in 2.1 but \bar{D} is unary which means $\bar{D} = \langle D_{\alpha,\eta} : \alpha < \mu, \eta \in \mathbf{T}_{< \lambda}^\alpha \rangle$ that is $\eta \in \prod_{\varepsilon < \zeta} \theta_{\alpha,\varepsilon}$ for some $\zeta < \lambda$, see 2.6(9), and if

$\theta_{\alpha,\ell g(\eta)} \geq \aleph_0$ then each $D_{\alpha,\eta}$ a $(\aleph_0 + |\varepsilon|^+)$ -complete filter on $\{\eta^\wedge \langle j \rangle : j < \theta_{\alpha,\ell g(\eta)}\}$.
 2) For such \mathfrak{r} and for $\bar{\eta} \in \mathbf{T}_{\subseteq \mu, \alpha}$ we let $D_{\bar{\eta}}$ be the set of $\mathcal{Y} \subseteq \text{suc}(\bar{\eta})$ such that for some function f with domain $\cup \{\text{suc}(\bar{\eta} \restriction \alpha) : \alpha \in \text{dom}(\bar{\eta})\}$ satisfying $\alpha \in \text{dom}(\bar{\eta}) \Rightarrow f(\bar{\eta} \restriction \alpha) \in D_{\alpha,\eta(\alpha)}$ we have $\mathcal{Y} = \{\bar{\nu} \in \text{suc}(\bar{\eta}) : \text{if } \alpha \in \text{dom}(\bar{\eta}) \text{ then } \nu_\alpha(\text{ht}(\bar{\eta})) \in f(\bar{\eta} \restriction \alpha)\}$.

3) We say \mathfrak{x} , a 2-ip, is non-trivial when no $D_{\langle \eta \rangle}$ is a principal ultrafilter; we may say $\bar{D}_{\mathfrak{x}}$ is non-trivial; similarly for 3-ip.

Claim 2.3. *If \mathfrak{x} is a 3-ip then \mathfrak{y} is 2-ip where \mathfrak{y} is like \mathfrak{x} replacing $\bar{D}_{\mathfrak{x}}$ by $\langle D_{\bar{\eta}} : \bar{\eta} \in \mathbf{T}_{\subseteq \mu, < \lambda} \rangle$ as defined as in Definition 2.2(2).*

Proof. We use the “just” in 2.1(F). □_{2.3}

Hypothesis 2.4. 1) We fix \mathfrak{x} a ι -ip where $\iota = 2$.

2) But we consider a 3-ip as a 2-ip, see 2.2(1), 2.3(1) so \mathfrak{x} may be derived from a 3-ip.

Convention 2.5. 1) Let κ, ∂ denote (strongly) inaccessible cardinals $\leq \lambda$.

2) Let u denote a subset of μ .

Notation 2.6. 0) In Definition 2.1, $\lambda = \lambda_{\mathfrak{x}}, \bar{\theta} = \bar{\theta}_{\mathfrak{x}} = \langle \theta_{\mathfrak{x}, \alpha, \varepsilon} : \alpha < \mu_{\mathfrak{x}}, \varepsilon < \lambda_{\mathfrak{x}} \rangle, \bar{\theta}_{\mathfrak{x}, \alpha} = \langle \theta_{\mathfrak{x}, \alpha, \varepsilon} : \varepsilon < \lambda_{\mathfrak{x}} \rangle, \bar{D} = \bar{D}_{\mathfrak{x}}$, etc.

1) For $u \subseteq \mu$ and $\alpha \leq \lambda$ let $\mathbf{T}_{u, \alpha} = \{ \bar{\eta} : \bar{\eta} = \langle \eta_{\varepsilon} : \varepsilon \in u \rangle \text{ and } \eta_{\varepsilon} \in \prod_{\beta < \alpha} \theta_{\varepsilon, \beta} \text{ for each } \varepsilon \in u \}$.

2) Let $\mathbf{T}_{\alpha}^{\kappa} = \cup \{ \mathbf{T}_{u, \alpha} : u \in [\mu]^{< \kappa} \}$, this is not as in §1; and let $\mathbf{T}^{\kappa} = \cup \{ \mathbf{T}_{\alpha}^{\kappa} : \alpha < \kappa \}$, $\mathbf{T} = \cup \{ \mathbf{T}^{\kappa} : \kappa \leq \lambda \}$; and $\mathbf{T}_{u, < \alpha} = \cup \{ \mathbf{T}_{u, \beta} : \beta < \alpha \}$ and $\mathbf{T}_{\subseteq u, \alpha} = \cup \{ \mathbf{T}_{v, \alpha} : v \subseteq u, |v| < \kappa \}$, etc., for $u \subseteq \mu, \alpha \leq \lambda$; we may write $\mathbf{T}[u, \alpha]$ instead $\mathbf{T}_{u, \alpha}$, etc.

3) For $\bar{\eta} \in \mathbf{T}_{u, \alpha}, \beta \leq \alpha$ and $v \subseteq u$ let

- (a) $\bar{\eta} \upharpoonright (v, \beta) = \langle \eta_{\varepsilon} \upharpoonright \beta : \varepsilon \in v \rangle$ and
- (b) $\bar{\eta} \upharpoonright \beta = \bar{\eta} \upharpoonright (u, \beta)$.

4)

- (a) If $\bar{\eta} \in \mathbf{T}_{u, \alpha}$ then let $u = \text{dom}(\bar{\eta})$ and $\alpha = \text{ht}(\bar{\eta})$
- (b) we define the partial order $\leq_{\mathbf{T}}$ on \mathbf{T} by $\bar{\eta} \leq_{\mathbf{T}} \bar{\nu}$ iff $\bar{\eta} = \bar{\nu} \upharpoonright (\text{dom}(\bar{\eta}), \text{ht}(\bar{\eta}))$.

5) If $\bar{\eta} \in \mathbf{T}_{u, \alpha}, u \subseteq \mu$ and $\alpha < \lambda$ then let $\text{suc}(\bar{\eta}) = \{ \bar{\nu} \in \mathbf{T}_{u, \alpha+1} : \bar{\nu} \upharpoonright \alpha = \bar{\eta} \}$.

6) Let $\mathbf{T}_{\subseteq u, < \kappa} = \{ \bar{\eta} : \text{dom}(\bar{\eta}) \in [u]^{< \kappa} \text{ and } \text{ht}(\bar{\eta}) = \alpha \}$ for $u \subseteq \mu$.

7) We say $\bar{\eta}, \bar{\nu} \in \mathbf{T}$ are compatible when $\alpha \in \text{dom}(\bar{\eta}) \cap \text{dom}(\bar{\nu}) \Rightarrow (\eta_{\alpha} \leq \nu_{\alpha}) \vee (\nu_{\alpha} \leq \eta_{\alpha})$.

8) We say \bar{u} represents $\mathbf{T}_{u, < \kappa}$ when:

- (a) $u \in [\mu]^{\leq \kappa}$
- (b) $\bar{u} = \langle u_{\zeta} : \zeta \in E \rangle$
- (c) E is a club of κ
- (d) $\langle u_{\zeta} : \zeta \in E \rangle$ is \subseteq -increasing continuous with union u
- (e) $|u_{\zeta}| < \kappa$ for $\zeta \in E$ and $u_0 = \emptyset$.

9) Let $\mathbf{T}_{\gamma}^{\alpha} = \prod_{\varepsilon < \gamma} \theta_{\alpha, \varepsilon}$ and $\mathbf{T}_{< \gamma}^{\alpha} = \cup \{ \mathbf{T}_{\varepsilon}^{\alpha} : \varepsilon < \gamma \}$.

10) For a \leq_T -directed $T \subseteq \mathbf{T}$ we say $\bar{\eta} = \lim(T)$ when:

- (a) $\text{dom}(\bar{\eta}) = \cup \{ \text{dom}(\bar{\eta} : \eta \in T) \}$
- (b) $\text{ht}(\bar{\eta}) = \cup \{ \text{ht}(\bar{\eta} : \eta \in T) \}$
- (c) for every $\alpha \in \text{dom}(\bar{\eta})$ we have $\eta_{\alpha} := \cup \{ \nu_{\alpha} : \bar{\nu} \in T \text{ is such that } \alpha \in \text{dom}(\bar{\nu}) \}$.

11) We may write $\langle \bar{\nu}_{\alpha} : \alpha < \alpha_* \rangle$ when $T = \{ \bar{\nu}_{\alpha} : \alpha < \alpha_* \}$.

Definition 2.7. By induction on $\kappa \leq \lambda$ for² any set $u \subseteq \mu$ of $\leq \kappa$ ordinals such that $\kappa \geq \sup\{\theta_{\alpha,\varepsilon}^+ : \alpha \in u, \varepsilon < \kappa\}$, (recall κ denotes a strongly inaccessible cardinal and u is of cardinality $\leq \kappa$); we define the forcing notion $\mathbb{Q}_{u,\kappa}^1$ and more as follows:

- (A) $p \in \mathbb{Q}_{u,\kappa}^1$ iff for some witnesses \bar{u}, S, Ξ we have (we may write $\bar{u} = \bar{u}_p = \langle u_\zeta : \zeta \in E \rangle = \langle u_{p,\zeta} : \zeta \in E_p \rangle, S_p, \Xi_p$, though pedantically, they are not unique iff
- (a) (α) p is a subset of $\cup\{\mathbf{T}_{w,\beta} : w \subseteq u, |w| < \kappa \text{ and } \beta < \kappa\}$
 - (β) \bar{u} represents $\mathbf{T}_{u,<\kappa}$
 - (γ) if $\zeta \in E := \text{dom}(\bar{u})$ then $\zeta = |u_\zeta|$
 - (δ) let $\text{Dom}(p) := \cup\{\text{dom}(\bar{\eta}) : \bar{\eta} \in p\}$
 - (b) if $\bar{\eta} \in p \cap \mathbf{T}_{v,\alpha}, v \subseteq w \in [u]^{<\kappa}$ and $\beta \in [\alpha, \kappa)$ then there is $\bar{\nu} \in p \cap \mathbf{T}_{w,\beta}$ such that $\bar{\eta} = \bar{\nu} \upharpoonright (w, \alpha)$
 - (c) p has a trunk $\bar{\eta} = \text{tr}(p)$, it is the unique $\bar{\eta}$ such that:
 - (α) $\bar{\eta} \in p$
 - (β) if $\bar{\nu} \in p$ and $\alpha \in \text{dom}(\bar{\nu}) \cap \text{dom}(\bar{\eta})$ then $(\eta_\alpha \leq \nu_\alpha) \vee (\nu_\alpha \leq \eta_\alpha)$
 - (γ) if $\text{ht}(\bar{\eta}) = 0$ then $\text{dom}(\bar{\eta}) = \emptyset$
 - (δ) if $\bar{\eta}'$ satisfies clauses $(\alpha), (\beta), (\gamma)$ then
 - $\text{ht}(\bar{\eta}') = \text{ht}(\bar{\eta})$
 - $\bar{\eta}' = \bar{\eta} \upharpoonright (\text{dom}(\bar{\eta}'), \text{ht}(\bar{\eta}))$
 - (d) (α) if $\bar{\eta} \in p \cap \mathbf{T}_{w,\alpha}$ and $v \subseteq w$ and $\beta \leq \alpha$ then $\bar{\eta} \upharpoonright (v, \beta) \in p$
 - (β) if $\text{tr}(p) \leq_{\mathbf{T}} \bar{\eta} \in p \cap \mathbf{T}_{v,\alpha}$ then $p \cap \text{suc}(\bar{\eta}) \in D_{v,\alpha}$
 - (e) $S \subseteq S_{\mathbf{T}} \cap \kappa \setminus E_p$ is disjoint to $(\text{ht}(\text{tr}(p)) + 1)$, and is nowhere stationary, (that is: if δ is a limit ordinal $\leq \kappa$ of uncountable cofinality then $S_p \cap \delta$ is not stationary in δ)
 - (f) (α) Ξ is a subset of $\Xi_{u,<\kappa} := \{\mathbf{y} : \mathbf{y} \text{ is of the form } (v, \partial, \Lambda) = (v_{\mathbf{y}}, \partial_{\mathbf{y}}, \Lambda_{\mathbf{y}}), \partial \in \kappa \cap S_p \subseteq S_{\mathbf{T}}, v \subseteq u, |v| \leq \partial, \Lambda \text{ is a set of } \leq \partial \text{ open dense subsets of } \mathbb{Q}_{v,\partial}^1\}$
 - (β) for $\mathbf{y} = (v, \partial, \Lambda) \in \Xi_p$ or $\Xi_{\mathbf{y},\kappa}$ let $\mathbf{T}_{\mathbf{y}} = \{\bar{\eta} \in \mathbf{T}_{v,\partial} : \text{for every } \mathcal{J} \in \Lambda \text{ there is } q \in \Lambda \text{ such that } \bar{\eta} \in \lim(q)\}$, see Clause (C) below noting $\partial < \kappa$ so the induction hypothesis apply to it
 - (γ) let $\Xi_{v,\partial} = \{\mathbf{y} : \text{for some } \Lambda, \mathbf{y} = (v, \partial, \Lambda) \text{ satisfies the demands above}\}$
 - (g) $\{(v, \partial, \Lambda) \in \Xi : \partial = \partial_*\}$ has at most ∂_* members for any $\partial_* \in S_p$
 - (h) if $\mathbf{y} \in \Xi_p$ and then $\partial_{\mathbf{y}} \notin E_p$, essentially follows by clause (f)
 - (i) if $\zeta \in E_p$ then u_ζ is (p, ζ) -big which means $\mathbf{y} \in \Xi_p \wedge \partial_{\mathbf{y}} \leq \zeta \Rightarrow v_{\mathbf{y}} \subseteq u_{p,\zeta}$; in fact $\zeta < \partial_{\mathbf{y}}$ follows.
 - (j) Assume $\bar{\eta} \in \mathbf{T}_{\subseteq u, < \kappa}$ and $\text{ht}(\bar{\eta})$ is a limit ordinal and let $\zeta = \zeta_{p,\bar{\eta}} := \min\{\varepsilon \in E_p : \varepsilon \geq \text{ht}(\bar{\eta}) \text{ and } u_{p,\varepsilon} \supseteq \text{dom}(\bar{\eta})\}$. Then $\bar{\eta} \in p$ iff for every $\mathbf{y} \in \Xi_p$ satisfying $\partial_{\mathbf{y}} \leq \zeta$ we have $\bar{\eta}$ is $\leq_{\mathbf{T}}$ -compatible with some member of $\Lambda_{\mathbf{y}}$.

²Note that $\mathbb{Q}_{u,\kappa}^1$ depends on the parameters from Definition 2.1 only up to κ actually on $\langle \theta_{\alpha,\varepsilon} : \alpha < \kappa, \varepsilon \in u \rangle$ and $\langle D_{\bar{\eta}} : \bar{\eta} \in \mathbf{T}_{\subseteq u, < \kappa} \rangle$ only.

- (B) $\mathbb{Q}_{u,\kappa}^1 \models "p \leq q" \text{ iff } p \supseteq q$ and, of course, $p, q \in \mathbb{Q}_{u,\kappa}^1$
- (C) for $p \in \mathbb{Q}_{u,\kappa}^1$ let
- $\lim(p) = \{\bar{\eta} \in \mathbf{T}_{u,\kappa} : \text{if } v \in [u]^{<\kappa} \text{ and } \alpha < \kappa \text{ then } \bar{\eta} \upharpoonright (u, \alpha) \in p\}$
 - $\text{Dom}(p) = u$, equivalently $\cup\{\text{dom}(\bar{\eta}) : \bar{\eta} \in p\} = u$, see (A)(b)
- (D) if $p \in \mathbb{Q}_{u,\kappa}^1$, $\alpha < \kappa$ and $v \subseteq \text{dom}(p)$ has cardinality $< \kappa$ and $\bar{\eta} \in \mathbf{T}_{v,\alpha} \cap p$ then $p^{[\bar{\eta}]} := \{\bar{\nu} \in p : \bar{\eta}, \bar{\nu} \text{ are compatible}\}$, (see 2.6(7), usually we assume $\text{ht}(\bar{\eta}) \geq \text{ht}(\text{tr}(p))$ and $\text{dom}(\eta) \supseteq \text{dom}(\text{tr}(p))$)
- (E) (a) for $\alpha \in u$ let $\eta_\alpha = \eta_{u,\kappa,\alpha}$ be the following $\mathbb{Q}_{u,\kappa}^1$ -name
- $$\cup\{(\text{tr}(p))_\alpha : p \in \mathbf{G}_{\mathbb{Q}_{u,v}^1} \text{ and } \alpha \in \text{dom}(\text{tr}(p))\}$$
- (b) $\bar{\eta} = \bar{\eta}_{u,\kappa}$ is $\langle \eta_\alpha : \alpha \in u \rangle$.

Remark 2.8. 1) We may make \bar{u}_p, S_p, Ξ_p and a club disjoint to S_p part of p .
 2) Note that for ζ an accumulation point of S_p there may be $\varepsilon < \zeta$ and $\bar{\eta} \in \mathbf{T}_{u,\varepsilon}$ such that $\bar{\eta} \notin p$ but $[\xi \in \zeta \cap S_p \Rightarrow \bar{\eta} \upharpoonright u_{p,\xi} \in p]$.
 3) Note that the demand 2.7(A)(b) is part of the definition, not proved as in §(1A), but we have to pay for this in checking $p \in \mathbb{Q}_{u,\kappa}^2$. Also in 2.7(A)(g) we do not require $\langle \cup\{v_{\mathbf{y}} : \mathbf{y} \in \Xi_p \text{ and } \partial_{\mathbf{y}} = \partial\} : \partial \in S \rangle$ is \subseteq -increasing continuous; it is not unreasonable to add this but it just means restricting ourselves to a dense subset.

Definition 2.9. Assume $\kappa \leq \lambda$ and $u \subseteq \mu$.

1) We define the forcing notion $\mathbb{Q}_{u,\kappa}^2$ as follows:

- (A) $p \in \mathbb{Q}_{u,\kappa}^2 \text{ iff } p \in \mathbb{Q}_{v,\kappa}^1 \text{ for some } v \in [u]^{\leq \kappa}$; so $\text{Dom}(p) = v$, see clause (A)(a)(δ) of Definition 2.7
- (B) $\mathbb{Q}_{u,\kappa}^2 \models "p_1 \leq p_2" \text{ iff } p_1, p_2 \in \mathbb{Q}_{u,\kappa}^2 \text{ and } \text{Dom}(p_1) \subseteq \text{Dom}(p_2) \text{ and } p_2 \cap \mathbf{T}_{\subseteq \text{Dom}(p_1), < \kappa} \subseteq p_1$.

2) For $\alpha \in u$ we define the $\mathbb{Q}_{u,\kappa}^2$ -name $\eta_\alpha = \eta_{u,\kappa,\alpha}$ as $\cup\{(\text{tr}(p))_\alpha : p \in \mathbf{G}_{\mathbb{Q}_{u,\kappa}^2} \text{ and } \alpha \in \text{dom}(\text{tr}(p))\}$.

3) Let $\bar{\eta} = \bar{\eta}_{u,\kappa} = \langle \eta_{u,\kappa,\alpha} : \alpha \in u \rangle$.

4) Let $\mathbb{Q}_{u,\kappa}^3$ be $\mathbb{Q}_{u,\kappa}^1$ when $1 \leq |u| \leq \kappa$ and $\mathbb{Q}_{u,\kappa}^2$ when $|u| > \kappa$.

Convention 2.10. 1) We may write $\mathbb{Q}_{u,\kappa}^\iota$ for $\iota \in \{1, 2, 3\}$ but if $\iota = 1$ we assume $|u| \leq \kappa$.

2) Not specifying ι means that it does not matter which $\iota \in \{1, 2, 3\}$ we use; e.g. in 2.14.

Claim 2.11. 1) In clause (A) of Definition 2.7 if $D_{\bar{\eta}} = \{\text{suc}(\bar{\eta})\}$ for every $\bar{\eta} \in \mathbf{T} \subseteq u, < \kappa$ then the tuple $(\kappa, \bar{u}_p, \text{tr}(p), \Xi_p)$ determine p .

2) If $p \in \mathbb{Q}_{u,\kappa}$, $\delta < \kappa$ and $\langle \bar{\eta}_i : i < \delta \rangle$ is $\leq_{\mathbf{T}}$ -increasing, see 2.6(4)(β), $\bar{\eta}_i \in p$ and recalling 2.7(A)(j) we have:

if $\cup\{\text{ht}(\bar{\eta}_i) : i < \delta\} \notin S_p$ or $\text{ht}(\bar{\eta}_i)$ is constant then the $\leq_{\mathbf{T}}$ -lub of $\langle \bar{\eta}_i : i < \delta \rangle$ belongs to p .

3) Clause (b) of 2.7(A) follows from the rest.

4) $\Vdash_{\mathbb{Q}_{u,\kappa}^\iota} "\eta \text{ is a member of } \prod_{i < \kappa} \theta_{\alpha,i} \subseteq {}^\lambda \lambda \text{ for } \alpha \in u$; also the sequence, in fact,

$\bar{\eta} = \langle \eta_\alpha : \alpha \in u \rangle$ is generic for $\mathbb{Q}_{u,\kappa}$.

5) The η_α 's ($\alpha \in u$) are $\mathbb{Q}_{u,\kappa}$ -names of new, pairwise distinct κ -reals provided that³:

³Those are natural sufficient conditions

- (a) if really \mathfrak{x} is a 3-ip and no $D_{\mathfrak{x},\alpha,\eta}$ is a principal ultrafilter
- (b) in general, if for every $\bar{\eta} \in \mathbf{T}$ and $\alpha \in \text{dom}(p)$ and $X \in D_{\mathfrak{x},\bar{\eta}}$ there are $\bar{\nu}_1, \bar{\nu}_2 \in X$ such that $\bar{\nu}_1 \upharpoonright \alpha = \bar{\nu} \upharpoonright \alpha$ but $\nu_{1,\alpha} \neq \nu_{2,\alpha}$.

Proof. 1),2) Read the definition particularly 2.7(A)(h).

3) Easy.

4),5) We prove this by induction on κ . □_{2.11}

Claim 2.12. Assume $\kappa \leq \lambda$ and $w \subseteq \mu$.

- 1) If $u \in [w]^{\leq \kappa}$ then $\mathbb{Q}_{u,\kappa}^1 \subseteq \mathbb{Q}_{u,\kappa}^2$ and if $u \subseteq w(\subseteq \mu_{\mathfrak{x}})$ then, $\mathbb{Q}_{u,\kappa}^2 \subseteq \mathbb{Q}_{w,\kappa}^2$.
- 2) If $p \in \mathbb{Q}_{u,\kappa}^2$, $\text{Dom}(p) \subseteq v \in [w]^{\leq \kappa}$ and $q = p^{+v} := \{\bar{\eta} \in \mathbf{T}_{\subseteq v,\kappa} : \bar{\eta} \upharpoonright \text{Dom}(p) \in p\}$ then $q \in \mathbb{Q}_{v,\kappa}^1 \subseteq \mathbb{Q}_{u,\kappa}^2$ and $\mathbb{Q}_{u,\kappa}^2 \models "p \leq q"$ and $\text{tr}(q) = \text{tr}(p)$.
- 3) If $u \subseteq v \in [w]^{\leq \kappa}$ then $p \in \mathbb{Q}_{u,\kappa}^2 \models "p \leq q" \Rightarrow \mathbb{Q}_{v,\kappa}^1 \models "p^{+v} \leq q^{+v}"$.
- 3A) In part (3), if $p, q \in \mathbb{Q}_{u,\kappa}^1$ then we have \Leftrightarrow .
- 4) If $u \in [\mu]^{\leq \kappa}$ then $\mathbb{Q}_{u,\kappa}^1$ is a dense open subset of $\mathbb{Q}_{u,\kappa}^2$.

Discussion 2.13. It seems natural to think that

- (*) $u \subseteq v \subseteq \mu \Rightarrow \mathbb{Q}_{u,\kappa}^2 \triangleleft \mathbb{Q}_{v,\kappa}^2$, equivalently
- (*) if $u \subseteq v \in [w]^{\leq \kappa}$, $w \subseteq \mu$ and \mathcal{J} is a predense subset of $\mathbb{Q}_{u,\kappa}^1$ or just of $\mathbb{Q}_{u,\kappa}^2$ then $\mathcal{J}^{+v} = \{p^{+v} : p \in \mathcal{J}\}$ is a predense subset of $\mathbb{Q}_{v,\kappa}^1$ and also of $\mathbb{Q}_{w,\kappa}^2$.

But a sufficient condition is that $\mathbb{Q}_{u,\kappa}^1$ is nice enough: if $\mathcal{J} \subseteq \mathbb{Q}_{u,\kappa}^1$ is dense open and $\bar{\eta} \in \mathbf{T}_{\subseteq u, < \kappa}$ then for some $p \in \mathbb{Q}_{u,\kappa}^1$ we have

- $\text{tr}(p) = \bar{\eta}$
- for some $\zeta \in E_p$, for every $\bar{\nu} \in \mathbf{T}_{u_p, \zeta, \zeta}$ we have $p^{[\bar{\nu}]} \in \mathcal{J}$.

Proof. Straight. □

Claim 2.14. Let $\kappa \leq \lambda$ and $u \subseteq \mu$.

- 0) $\mathbb{Q}_{u,\kappa}$ has cardinality $\leq |u|^\kappa$, in fact $= (|u| + 2)^\kappa$ if $u \neq \emptyset$.
- 1) If $|u| \leq \kappa$ then $\mathbf{T}_{\subseteq u, < \kappa}$ belongs to $\mathbb{Q}_{u,\kappa}^1$ and $\mathbb{Q}_{u,\kappa}^1 \models "\mathbf{T}_{\subseteq u, < \kappa} \leq p"$ if $p \in \mathbb{Q}_{u,\kappa}^1$.
- 2) If $p \in \mathbb{Q}_{u,\kappa}$ and $\alpha < \kappa$ and v is a (p, ∂) -big, i.e. $v \subseteq \text{dom}(p)$ has cardinality ∂ and $[(u', \kappa', \Lambda) \in \Xi_p \wedge \kappa' \leq \partial \Rightarrow u' \subseteq v]$ pedantically this holds for some witness (\bar{u}_p, S_p, Ξ_p) of p then $\{p^{[\bar{\eta}]} : \bar{\eta} \in \mathbf{T}_{v,\alpha} \cap p\}$ is predense above p in $\mathbb{Q}_{u,\kappa}$.
- 3) If $p \in \mathbb{Q}_{u,\kappa}^\iota$, $v \subseteq u$, $\text{ht}(\text{tr}(p)) < \kappa_1 \leq \kappa$ then
 - $p \cap \mathbf{T}_{\subseteq v, < \kappa_1} \in \mathbb{Q}_{v,\kappa_1}^\iota$ and
 - if $\iota = 2$ then $\mathbb{Q}_{u,\kappa}^2 \models "(p \cap \mathbf{T}_{\subseteq v, < \kappa}) \leq p"$.

4) $\mathbb{Q}_{u,\kappa}$ satisfies the κ^+ -c.c. (even $*_{S_{\kappa^+}^1}$ (see [Sh:546])), moreover is essentially κ -centered.

5)

- (a) if $\alpha \in u$ and $u \subseteq \mu$, then $\Vdash_{\mathbb{Q}_{u,\kappa}} " \eta_{u,\kappa,\alpha} \in \prod_{\varepsilon < \kappa} \theta_{\alpha,\varepsilon} "$
- (b) $\bar{\eta} := \lim\{\text{tr}(p) : p \in \mathbf{G}_{\mathbb{Q}_{u,\kappa}}\}$ see 2.7(E), is a member of $\mathbf{T}_{u,\kappa} := \{\langle \eta_\alpha : \alpha \in u \rangle : \text{if } \alpha \in u \text{ then } \eta_\alpha \in \prod_{\varepsilon < \kappa} \theta_{\alpha,\varepsilon}\}$, i.e. a $\mathbb{Q}_{u,\kappa}$ -name of a member.

6) $\langle \eta_\alpha : \alpha \in u \rangle$ is a generic for $\mathbb{Q}_{u,\kappa}$.

Proof. Straight. □

Claim 2.15. Assume $\kappa \leq \lambda, u \subseteq \mu$.

0) The conditions $p, q \in \mathbb{Q}_{u, \kappa}^1$ are compatible iff for some $\partial < \kappa$ and v which is (p, ∂) -big, (q, ∂) -big and $\bar{\eta} \in p \cap q$ we have $\text{dom}(\bar{\eta}) = v, \text{ht}(\eta) = \partial$ and $\text{tr}(p) \leq_{\mathbf{T}} \bar{\eta}, \text{tr}(q) \leq_{\mathbf{T}} \eta$.

1) If $|u| \leq \kappa$ and $p_i \in \mathbb{Q}_{u, \kappa}^1$ for $i < i(*)$, $\bar{v} \in \cap \{p_i : i < i(*)\}$ and $i < i(*) \Rightarrow \text{tr}(p_i) \leq_{\mathbf{T}} \bar{v} \in p_i$, e.g. $i < i(*) \Rightarrow \bar{v}_i = \bar{v}$ and $i(*) \leq \text{ht}(\bar{v})$ then $p := \cap \{p_i : i < i(*)\}$ is a $\leq_{\mathbb{Q}_{u, \kappa}^1}$ -lub of $\{p_i : i < i(*)\}$ in $\mathbb{Q}_{u, \kappa}^1$.

2) If $p_i \in \mathbb{Q}_{u, \kappa}^2$ for $i < i(*)$ and $\bar{v} \in \mathbf{T}_{\subseteq u, \kappa}$ and $i < i(*) \Rightarrow \text{tr}(p_i) \leq \bar{v}$ and $i < i(*) \Rightarrow \bar{v} \restriction \text{Dom}(p_i) \in p_i$ then:

- $\{p_i : i < i(*)\}$ has a common upper bound in $\mathbb{Q}_{u, \kappa}^2$
- letting $v = \cup \{\text{Dom}(p_i) : i < i(*)\}$ we have $p := \cap \{(p_i^{+v})^{[\bar{v}]} : i < i(*)\}$ is a $\leq_{\mathbb{Q}_{u, \kappa}^2}$ -ub of $\{p_i : i < i(*)\}$.

3) If $p_i \in \mathbb{Q}_{u, \kappa}^1$ for $i < \delta$ is $\leq_{\mathbb{Q}_{u, \kappa}^2}$ -increasing, δ a limit ordinal $< \kappa$ then $\bar{v} := \lim \langle \text{tr}(p_i) : i < \delta \rangle$ is as required in part (1) (hence $\langle p_i : i < \delta \rangle$ has a common upper bound) provided that $i < \delta \Rightarrow \text{ht}(\bar{v}) \notin S_{p_i}$ noting $\text{ht}(\bar{v}) = \sup \{\text{ht}(\text{tr}(p_i)) : i < \delta\}$.

4) If $p_i \in \mathbb{Q}_{u, \kappa}^2$ for $i < \delta$ is $\leq_{\mathbb{Q}_{u, \kappa}^2}$ -increasing, $\delta < \kappa$ limit ordinal then $\bar{v} = \text{tr} \langle \text{tr}(p_i) : i < \delta \rangle$ is as required in part (2) when $i < \delta \Rightarrow h(\bar{v}) \notin S_p$.

Proof. 0) First, why the first conditions imply the second. Assume p, q have a common upper bound, in $\mathbb{Q}_{u, \kappa}^1$ of course. We may choose $v \in [u]^{< \kappa}$ which is (p, ∂) -big for some ∂ , in fact any $\partial \in E_p$ will do and let $\bar{v} \in p$ be such that $\text{dom}(\bar{v}) = v, \text{ht}(\bar{v}) = \partial$ clearly \bar{v} is as required; really $\text{tr}(r)$ should do.

Second, why the second conditions implies the third? So assume $\bar{v} \in T_{u, < \kappa}$ is as required, then let $r = p^{[\bar{v}]} \cap q^{[\bar{v}]}$. Now for proving $r \in \mathbb{Q}_{u, \kappa}$, in checking recall that (b) of Definition 2.7(A) holds by claim 2.11(3) and we let

$$E_r = \{\delta < \kappa : u_{p, \delta} = u_{q, \delta} \text{ and so } \delta \in E_p \cap E_q\}$$

$$\bar{u} = \bar{u}_p \restriction E_r = \bar{u}_q \restriction E_r$$

$$S_r = S_p \cup S_q$$

$$\Xi_r = \{\mathbf{y} : \mathbf{y} \in \Xi_p \cup \Xi_q \text{ and } \partial_{\mathbf{y}} > \text{ht}(\bar{v})\}.$$

Of course, $\mathbb{Q}_{u, \kappa}^1 \models "p \leq r \wedge q \leq r"$ as this means $r \subseteq p, r \subseteq q$ which holds as $r \subseteq p^{[\bar{v}]} \subseteq p$ and $r \subseteq q^{[\bar{v}]} \subseteq q$. So we are done.

1) Similarly to part (0), only now we have to note then $\partial \in S_r \Rightarrow |\Xi_r \cap \Xi_{u, \partial}| \leq \partial$ as the union of $\leq i(*) + \partial \leq \partial$ such sets.

2) Similarly.

3) Clearly $i_1 < i_2 < \delta \Rightarrow p_{i_1} \leq_{\mathbb{Q}_{u, \kappa}^1} p_{i_2} \Rightarrow \text{tr}(p_{i_1}) \in p_{i_2} \wedge \text{tr}(p_{i_1}) \leq_{\mathbf{T}} \text{tr}(p_{i_2})$, hence $\bar{v} = \lim \langle \text{tr}(p_i) : i < \delta \rangle$ is well defined and as $\delta < \kappa$ clearly $\bar{v} \in \mathbf{T}_{u, \kappa}$. For $i < \delta$, as $\{\text{tr}(p_j) : j \in [i, \delta)\} \subseteq p_i$ and $\langle \text{tr}(p_i) : i < \delta \rangle$ is $\leq_{\mathbf{T}}$ -increasing with limit \bar{v} and $\text{ht}(\bar{v}) \notin S_{p_i}$, clearly $\bar{v} \in p_i$. So $\bar{v} \in \cap \{p_i : i < \delta\}$ as promised.

4) Similarly. \square

Definition 2.16. 1) Assume $u \subseteq \mu$ and $\kappa \leq \lambda$.

Let $\mathbf{S}_{u,\kappa,<\gamma}^{\text{inc}} = \cup\{\mathbf{S}_{u,\kappa,\beta}^{\text{inc}} : \beta < \gamma\}$ where $\mathbf{S}_{u,\kappa,\gamma}^{\text{inc}}$ for $\gamma < \kappa$ is the set of sequences $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha, \bar{u}_\alpha, S_\alpha, \Xi_\alpha) : \alpha < \gamma \rangle$ such that⁴:

- (a) $p_\alpha \in \mathbb{Q}_{u,\kappa}$
- (b) $q_\alpha \in \mathbb{Q}_{u,\kappa}$
- (c) $q_\alpha \leq p_\beta$ for $\alpha < \beta < \gamma$
- (d) $\bar{u}_\alpha = \langle u_{\alpha,\varepsilon} : \varepsilon \in E_\alpha \rangle, S_\alpha, \Xi_\alpha$ witness $p_\alpha \in \mathbb{Q}_{u,\kappa}$ so e.g. E_α is a club of κ disjoint to S_{p_α}
- (e) $\mathbb{Q}_{u,\kappa} \models "p_\alpha \leq q_\alpha"$
- (f) $\text{ht}(\text{tr}(q_\alpha)) \geq \alpha$ and $\text{dom}(\text{tr}(q_\alpha))$ is (p_α, α) -big
- (g) $\text{ht}(\text{tr}(q_\alpha)) \in \cap\{E_\beta : \beta < \alpha\}$.

2) If $\delta < \kappa$ and $\bar{p} = \langle p_i : i < \delta \rangle$ is $\leq_{\mathbb{Q}_{u,\kappa}}$ -increasing and then $p = \lim(\bar{p})$ is defined by:

- $\text{dom}(p_\delta) = \cup\{\text{dom}(p_\alpha) : \alpha < \delta\}$
- $\bar{\eta} \in p$ iff $\bar{\eta} \in \mathbf{T}_{\subseteq u, < \kappa}$, $\text{dom}(\bar{\eta}) \subseteq \text{dom}(p_\delta)$ and $\forall \alpha < \delta \Rightarrow \bar{\eta} \upharpoonright (\text{dom}(\bar{\eta}) \cap \text{dom}(p_\alpha), \text{ht}(\bar{\eta}))$ belongs to p_α .

Claim 2.17. 0) In Definition 2.16(2) if $p \in \mathbb{Q}_{u,\kappa}$ then p is a $\leq_{\mathbb{Q}_{u,\kappa}}$ -lub of \bar{p} .

- 1) For every $p \in \mathbb{Q}_{u,\kappa}$ the sequence $\langle (p, p, \kappa) \rangle$ belongs to $\mathbf{S}_{u,\kappa,1}^{\text{inc}} \subseteq \mathbf{S}_{u,\kappa,<\kappa}^{\text{inc}}$.
- 2) $\mathbf{S}_{u,\kappa,<\kappa}^{\text{inc}}$ is closed under unions of \leq -increasing chains of length $< \kappa$.
- 3) If $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha, \bar{u}_\alpha, S_\alpha, \Xi_\alpha) : \alpha < \beta \rangle \in \mathbf{S}_{u,\kappa}^{\text{inc}}$ and $\beta < \kappa$ is non-zero then for some $p_\beta \in \mathbb{Q}_{u,\kappa}$ and E_β we have:

- if $\mathbb{Q}_{u,\kappa} \models "p_\beta \leq q_\beta"$ and $E \subseteq \cap\{E_\alpha : \alpha < \beta\}$ is a club of κ disjoint to $S_{q_\beta} \setminus (\beta + 1)$ then $\bar{\mathbf{x}}^\wedge \langle (p_\beta, q_\beta, E_\beta) \rangle \in \mathbf{S}_{u,\kappa,<\kappa}^{\text{inc}}$.

Proof. Straight, e.g. in part (0) use 2.14(4) and in (3), $p_\beta = q_{\beta-1}$ if β is a successor, p_β . □_{2.17}

Conclusion 2.18. $\mathbb{Q}_{u,\kappa}$ is κ -strategically complete.

Definition 2.19. For $\kappa \leq \lambda, u \subseteq \mu$ for $\gamma \leq \kappa$ we let $\mathbf{S}_{u,\kappa,<\gamma}^{\text{pr}} = \cup\{\mathbf{S}_{u,\kappa,\beta}^{\text{pr}} : \beta < \gamma\}$ and may write $\mathbf{S}_{u,\kappa,<\partial}^{\text{pr},\iota}, \mathbf{S}_{u,\kappa,\gamma}^{\text{pr},\iota}$ where for $\gamma \leq \kappa$, $\mathbf{S}_{u,\kappa,\gamma}^{\text{pr}}$ is the set of sequences of $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha, \bar{u}_\alpha, S_\alpha, \Xi_\alpha, \Upsilon_\alpha) : \alpha < \gamma \rangle$ (so $p_\alpha[\mathbf{x}] = p_\ell$, etc.) satisfying:

- (a) – (e) as in 1.20(3), (but not clauses (f),(g)!)
 - (f) (α) $\Upsilon_\alpha = \Upsilon(\alpha) < \kappa$ is increasing and let $\Upsilon_{<\alpha} = \cup\{\Upsilon_\beta : \beta < \alpha\}$
 - (β) $\Upsilon_\alpha > \text{ht}(\text{tr}(p_\alpha)) = \text{ht}(\text{tr}(q_\alpha))$
 - (γ) $\Upsilon_\beta \in E_\alpha$ when $\alpha < \beta$
 - (δ) $u_{\alpha,\Upsilon(\alpha)}$ is $(p_\alpha, \Upsilon_\alpha)$ -big for α successor
 - (g) if $\alpha < \beta < \gamma$ then
 - $q_\alpha \cap \mathbf{T}[u_{\alpha,\Upsilon(\alpha)}, \Upsilon_\alpha + 1] \subseteq p_\beta \cap q_\beta$
 - $E_\beta \cap (\Upsilon_\alpha + 1) = E_\alpha \cap (\Upsilon_\alpha + 1)$
 - $\bar{u}_\beta \upharpoonright (\Upsilon_{\alpha+1}) = \bar{u}_\alpha \upharpoonright (\gamma_\alpha + 1)$

⁴may add: if $\delta < \gamma$ is a limit ordinal then $p_\delta = \lim\langle p_\alpha : \alpha < \delta \rangle$ that is a limit ordinal, $\langle p_i : i < \delta \rangle$ is $\leq_{\mathbb{Q}_{u,\kappa}}$ -increasing and $\lim(\text{tr}(p_\ell) : i < \delta)$.

(h) if $\delta < \gamma$ is a limit ordinal then $p_\delta = \lim \langle p_\alpha : \alpha < \delta \rangle$.

Claim 2.20. Assume $u \subseteq \mu, \kappa \subseteq \lambda$.

1) For every $p \in \mathbb{Q}_{u,\kappa}$ the sequence $\langle (p, p, \kappa) \in \mathbf{S}_{u,\kappa}^{\text{pr}}$.
 2) If $\gamma < \kappa$ and $\bar{\mathbf{x}} = \langle (p_\alpha, q_\alpha, E_\alpha, \bar{u}_\alpha, S_\alpha, \Xi_\alpha, \Upsilon_\alpha) : \alpha \leq \gamma \rangle \in \mathbf{S}_{u,\kappa,\gamma+1}^{\text{pr}}$ then there are $p, E, \bar{u}, S, \Xi, \Upsilon$ such that

- $p \in \mathbb{Q}_{u,\kappa}$
- $p \cap \mathbf{T}_{\leq \Upsilon_\gamma} = p_\gamma \cap \mathbf{T}_{\leq \Upsilon_\gamma}$
- if $\mathbb{Q}_{u,\kappa} \models "p \leq q"$ and $q \cap \mathbf{T}_{\leq \Upsilon_\alpha} = p \cap \mathbf{T}_{\leq \partial_n}$ and $\Upsilon_\gamma < \Upsilon \in E_q \cap E$ (so E_q is part of a witness for q) then $\hat{\mathbf{x}} \langle (p, q, \bar{u}, S, \Xi, \Upsilon) \rangle \in \mathbf{S}_{u,\kappa,\leq \kappa}^{\text{pr}}$.

3) The union on a \triangleleft -increasing sequence of members of $\mathbf{S}_{u,\kappa}^{\text{pr}}$ of length $\leq \kappa$ belongs to $\mathbf{S}_{u,\kappa,\leq \kappa}^{\text{pr}}$.

3A) If $\langle \bar{\mathbf{x}}_\beta : \beta < \delta \rangle$ is \triangleleft -increasing, so $\bar{\mathbf{x}}_\beta = \langle (p_\alpha, q_\alpha, E_\alpha, \bar{u}_\alpha, S_\alpha, \Xi_\alpha, \Upsilon_\alpha) : \alpha < \gamma_\beta \rangle \in \mathbf{S}_{u,\kappa}^{\text{pr}}$ and $\langle \gamma_\beta : \beta < \delta \rangle$ is \leq -increasing and $\gamma = \cup \{ \gamma_\beta : \beta < \delta \} < \kappa$ then $\langle (p_\alpha, q_\alpha, E_\alpha, \bar{u}_\alpha, S_\alpha, \Xi_\alpha, \Upsilon_\alpha) : \alpha < \gamma \rangle \in \mathbf{S}_{u,\kappa,\gamma}^{\text{pr}}$.

3B) If in (2A), $\gamma = \kappa$ then $p_\kappa = \cap \{ p_\alpha : \alpha < \kappa \}$ belongs to \mathbb{Q}_κ and is a $\leq_{\mathbb{Q}_\kappa}$ -lub of $\{ p_\alpha, q_\alpha : \alpha < \kappa \}$.

Proof. Straight.

Now at last we deal with the λ -bounding. □

Crucial Claim 2.21. If $\kappa = \lambda$ or just $\kappa \in S^{\text{r}}$ and $u \subseteq \mu, \gamma < \kappa, \bar{\mathbf{x}} \in \mathbf{S}_{\kappa,\gamma+1}^{\text{inc}}$ and τ is a $\mathbb{Q}_{u,\kappa}^\iota$ -name of a member of \mathbf{V} then we can find $\mathbf{y} = (p_{\gamma+1}, q_{\gamma+1}, E_{\gamma+1}, \bar{u}_{\gamma+1}, S_{\gamma+1}, \Xi_{\gamma+1}, \Upsilon_{\alpha+1})$ such that

- (a) $\bar{\mathbf{x}} \hat{\ } \langle \mathbf{y} \rangle \in \mathbf{S}_{u,\gamma+2}^{\text{pr}}$
- (b) $\bar{\eta} \in q_{\gamma+1} \cap \mathbf{T}[u_{\gamma+1}, \Upsilon(\gamma+1), \Upsilon(\gamma+1)]$ then $q_{\gamma+1}^{[\bar{\eta}]}$ forces a value to τ .

Proof. Let $\mathcal{Y} = \{ \text{tr}(p) : p \in \mathbb{Q}_{u,\kappa}^\iota \text{ forces a value to } \tau \text{ and } h(\text{tr}(p)) > \min(E_\alpha) \}$ and for $\bar{\eta} \in \mathcal{Y}$ let $p_{\bar{\eta}}^*$ exemplify $\bar{\eta} \in \mathcal{Y}$, i.e.

- (*) $\text{tr}(p_{\bar{\eta}}^*) = \bar{\eta}$ and $p_{\bar{\eta}}^*$ forces a value to τ .

Clearly

- (*) (a) $\mathcal{Y} \subseteq \mathbf{T}_{\subseteq u, < \kappa}$
- (b) if $p \in \mathbb{Q}_{u,\kappa}^\iota$ then for some $\eta \in \mathcal{Y}$ we have $\text{tr}(p) \leq_{\mathbf{T}} \bar{\eta} \in p$.

As τ is a $\mathbb{Q}_{u,\kappa}^\iota$ -name of a member of \mathbf{V} there is a maximal antichain \mathcal{J} of $\mathbb{Q}_{u,\kappa}$ such that $p \in \mathbf{V} \Rightarrow p$ forces a value to τ . By 2.14(3) we have $|\mathcal{J}| \leq \kappa$ and let u_* be such that

- (*)₁ • $u_* \in [\mu]^{\leq \kappa}$
- if $\bar{v} \in \mathcal{Y} \cap \mathbf{T}_{\subseteq u_*, < \kappa}$ then $\text{dom}(p_{\bar{v}}^*) \subseteq u_*$
- $\text{Dom}(q_\gamma) \subseteq u_*$
- $u_* \supseteq \cup \{ \text{Dom}(p) : p \in \mathcal{J} \}$. Let p' be $q_{\gamma+1}^{[u_*]} = \{ \bar{\eta} \in \mathbf{T}_{\subseteq u_*, < \kappa} : \bar{\eta} \upharpoonright \text{dom}(p_\gamma) \in p_\gamma \}$ so $p \leq p' \in \mathbb{Q}_{u_*, \kappa}^1$. Clearly $\{ q \in \mathbb{Q}_{u_*, \kappa}^\iota : q \text{ forces (for } \Vdash_{\mathbb{Q}_{u,\kappa}} \text{) a value to } \tau \}$ is a dense open subset of $\mathbb{Q}_{u_*, \kappa}^\iota$.

By the Hypothesis 2.1, see 1.12(A) or 1.12(C) the weak compactness, there are $\partial \in S_{\mathfrak{x}} \cap \kappa$ and $u_{**} \in [u_*]^{\leq \partial}$ such that letting $\mathcal{Y}_\partial = \mathbf{Y} \cap \mathbf{T}_{\subseteq u_{**}, < \partial}$ we have

- (*) (a) $\bar{v} \in \mathcal{Y}_\partial \Rightarrow \ell g(\text{tr}(p_{\bar{v}})) < \partial$
- (b) if $p \in \mathbb{Q}_{u_{**}, \partial}$ then $\{\bar{\eta} : \text{tr}(p) \leq_{\mathbf{T}} \bar{\eta} \in p\} \cap \mathcal{Y}_\partial \neq \emptyset$
- (c) if $\bar{\eta} \in \mathcal{Y} \cap \mathbf{T}_{\subseteq u_{**}, < \partial}$ then $\partial \in E_{p_{\bar{\eta}}}^*$
- (d) $\{p_{\bar{\eta}}^* \cap \mathbf{T}_{\subseteq u_{**}, < \partial} : \bar{\eta} \in \mathcal{Y}_\partial\}$ is a maximal antichain of $\mathbb{Q}_{u_{**}, \partial}$.

Define:

- $p_{\gamma+1} = q_\gamma$
- $q_{\gamma+1} = \{\bar{\eta} \in p_{\gamma+1} : \text{if } \text{ht}(\bar{\eta}) \geq \partial \text{ and } \text{dom}(\bar{\eta}) \supseteq u_{**} \text{ then } \bar{\eta} \upharpoonright (u_{**}, \partial) \in \lim(p_{\bar{v}}^*) \text{ for some } \bar{v} \in \mathcal{Y}_\partial\}$
- $E_{\gamma+1} \subseteq E_\gamma \setminus (\Upsilon_{\gamma+1}, \partial + 1)$ is disjoint to $S_{p_{\bar{\eta}}}^*$ for some witness $(\bar{\eta}, S_{p_{\bar{\eta}}}^*, \Xi_{p_{\bar{\eta}}}^*)$ for $p_{\bar{\eta}}^*$ for every $\bar{\eta} \in q_{\gamma+1} \cap \mathcal{Y}_\partial$
- $\Upsilon_{\gamma+1} = \partial$.

□_{2.21}

Conclusion 2.22. *If $p \in \mathbb{Q}_{u, \kappa}$ and $p \Vdash \text{"}\mathcal{T}_\varepsilon \in \mathbf{V}\text{"}$ for $\varepsilon < \kappa$ then there are q and $\langle u_\varepsilon, \gamma_\varepsilon : \varepsilon < \lambda \rangle$ such that*

- $\mathbb{Q}_{u, \kappa} \models \text{"}p \leq q\text{"}$
- $\text{tr}(q) = \text{tr}(p)$
- $u_\varepsilon \subseteq \text{Dom}(p)$ has cardinality $< \kappa$ for $\varepsilon < \kappa$
- $\gamma_\varepsilon < \kappa$ for $\varepsilon < \kappa$
- if $\varepsilon < \kappa, \bar{\eta} \in \mathbf{T}_{u_\varepsilon, \gamma_\varepsilon} \cap q$ then $q[\bar{\eta}]$ belongs to $\mathbb{Q}_{u, \kappa}$ and forces a value to \mathcal{T}_ε .

Proof. Should be clear. □

Claim 2.23. 1) *If $\kappa \in S^{\mathfrak{x}}$ (if $S_{\mathfrak{x}}$ is the set of strongly inaccessible $\leq \lambda$ then this means $\kappa \leq \lambda$ is weakly compact) and $u \subseteq \mu$ then $\mathbb{Q}_{u, \kappa}$ is λ -bounding (as well as λ -strategically closed and λ^+ -c.c.), has cardinality $|u|^\kappa$ and add u pairwise distinct members of ${}^\lambda \lambda$ if \mathfrak{x} is non-trivial).*

2) *If $p \Vdash_{\mathbb{Q}_{u, \kappa}} \text{"}f : \kappa \rightarrow \mathbf{V}\text{"}$ and $\gamma < \kappa$ then there is q such that:*

- (a) $\mathbb{Q}_{u, \kappa} \models \text{"}p \leq q\text{"}$
- (b) $q \cap \mathbf{T}_{\subseteq \text{Dom}(p), \leq \gamma} = p \cap \mathbf{T}_{\subseteq \text{Dom}(p), \leq \gamma}$
- (c) for every $\zeta < \kappa$ for some $\partial < \kappa$, if v is (q, ζ) -big and $\bar{\eta} \in p \cap \mathbf{T}_{v, \zeta}$ then $q[\bar{\eta}]$ forces a value to $f(\zeta)$.

Proof. 1) By (2) and 2.18 and 2.14(3).

2) By 2.21. □_{2.23}

Claim 2.24. *Assume $\kappa \in S^{\mathfrak{x}}$ and (for 1), 2), 3)) $u \subseteq v \subseteq \mu$.*

0) *If $u \in [\mu]^{\leq \kappa}$ and Λ is a set of $\leq \kappa$ dense open subsets of $\mathbb{Q}_{u, \kappa}^1$ then for every $\bar{\eta} \in \mathbf{T}_{\subseteq u, < \kappa}$ there is $p \in \mathbb{Q}_{u, \kappa}$ with $\text{tr}(p) = \bar{\eta}$ and witness (\bar{u}, S, Ξ) for $p \in \mathbb{Q}_{u, \kappa}$ such that for every $\mathcal{I} \in \Lambda$ there is $\partial \in S, \partial > \text{ht}(\bar{\eta})$ such that*

- if $\bar{v} \in p \cap \mathbf{T}_{u_\partial, \partial}$ then $p[\bar{v}] \in \mathbf{T}$.

- 1) $\mathbb{Q}_{u,\kappa} \subseteq \mathbb{Q}_{v,\kappa}$ moreover $\mathbb{Q}_{u,\kappa} \subseteq_{\text{ic}} \mathbb{Q}_{v,\kappa}$.
- 2) For every $p \in \mathbb{Q}_{v,\kappa}$ there is $q \in \mathbb{Q}_{u,\kappa}$ such that: $\mathbb{Q}_{u,\kappa} \models "q \leq r" \Rightarrow p, r$ are compatible in $\mathbb{Q}_{v,\kappa}$.
- 3) $\mathbb{Q}_{u,\kappa} \triangleleft \mathbb{Q}_{v,\chi}$.

Proof. 0) Easy.

- 1) Easy by the amount of strategic completeness we have, see below. Obvious.
- 2) Let $\bar{v} = \text{tr}(p)$ and let $S = S_p, \Xi = \Xi_p$ witness p . Let

- (a) $\bar{v}' = \bar{v} \upharpoonright (\text{dom}(\bar{v}) \cap v, \text{ht}(\bar{v}))$
- (b) for $\mathbf{y} = (\partial, w, \Lambda) \in \Xi$ let $w' = w \cap u$ and choose Λ' as in part (0) with $(\partial, w, w', \Lambda, \Lambda')$ here playing the role of $(\kappa, v_1, u_2, \Lambda_1, \Lambda_2)$ there and let $\Xi' = \{\mathbf{y}' : \mathbf{y} \in \Xi\}$.

Let $q \in \mathbb{Q}_{u,\kappa}$ be defined by \bar{v}', Ξ' , see 2.11(1).

Now check.

- 3) Follows by (1)+(2).

□_{2.24}

§ 3. THE IDEAL: FOR THE LESS SET THEORETIC AUDIENCE

Our original aim was to disprove the existence of a forcing notion for λ having the properties of random real forcing. Having constructed one raises hopes for generalizing independence results about reals to ${}^\lambda 2$, so deriving independence results on λ -cardinality invariants. But in this section we try to get analysis for this ideal per se.

We shall try systematically to go over basic properties of the null ideal (and its relation with the meagre ideal).

The case of $\mathbb{Q}_{\bar{g}}$ is similar and we intend to comment on it in Part II. On the meagre and null ideals (for $\lambda = \aleph_0$) see Oxtoby [Oxt80]. On the measure algebra and random reals see Fremlin's book [Fre84] and web-cite.

How do we measure success? The main properties of the null ideal which comes to my mind are:

- ⊞ (a) an \aleph_1 -complete ideal
- (b) the quotient Boolean Algebra satisfies the c.c.c., i.e. there is no uncountable family of non-null pairwise disjoint Borel sets
- (c) the forcing is bounding: this means the quotient Boolean Algebra is (\aleph_0, ∞) -distributivity, that is if for each n , $\langle B_{n,k} : k \in \mathbb{N} \rangle$ is a Borel partition of a positive Borel set B then for some function $f : \mathbb{N} \rightarrow \mathbb{N}$, the set $\bigcap_n \bigcup_{k < f(n)} B_{n,k}$ is not null.

A priori, for the set theoretic purposes, generalizing (a), (b), (c) was the aim. But for the ideal itself probably more prominent then (c) and very nice itself, is

- (d) Fubini theorem: for a Borel set $A \subseteq [0, 1] \times [0, 1]$ we have:
 for all but null many x ,
 for all but null many y , $(x, y) \in A$ iff for all but null many y
 for all but null many x , $(x, y) \in A$.

Maybe it is helpful to stress

- ⊞ we are looking for λ^+ -complete, λ^+ -c.c., ideal with no atoms.

Below we make a list of statements generalizing the null ideal case, including the natural analogs of the properties listed above delaying a try on some further properties.

A reader who goes first to this section can note just that

- ⊕ \mathbb{Q}_λ is a set of (λ -closed) subtrees of ${}^\lambda 2$, parallel to the closed subsets of $[0, 1]_{\mathbb{R}}$ with positive Lebesgue measure partially ordered by inverse inclusion.

Definition 3.1. 1) For λ inaccessible, let $\text{id}(\mathbb{Q}_\lambda) = \{A \subseteq {}^\lambda 2 : \text{there are } i(*) \leq \kappa \text{ and dense open subsets } \mathcal{S}_i \text{ of } \mathbb{Q}_\lambda \text{ for } i < i(*) \text{ such that } \eta \in A \wedge i < i(*) \Rightarrow \eta \text{ fulfill } \mathcal{S}_i\}$, where:

- 2) η fulfills \mathcal{S} means $(\exists q \in \mathcal{S}_i)(\eta \in \lim_\lambda(q))$.
- 3) A λ -real is $\eta \in {}^\lambda 2$.

Convention 3.2. While §2 generalizes §1, in fact even for $\mu = 1$ there is a small difference. The one in §2 (for one λ -real) is better intrinsically, being preserved under permutation of κ . The one in §1 is more natural in the inductive proof. Usually it does not matter, hence we call them $\mathbb{Q}'_\lambda, \mathbb{Q}''_\lambda$ or \mathbb{Q}_λ when it does not matter. The ideal is $\text{id}(\mathbb{Q}'_\lambda), \text{id}(\mathbb{Q}''_\lambda)$ or $\text{id}(\mathbb{Q}_\kappa)$, but usually we write the proof for \mathbb{Q}'_λ .

Convention 3.3. $\lambda, \partial, \kappa$ vary on inaccessibles.

We have consulted several people on additional properties to be examined with them: A. Roslanowski ((Q) of first list, (B),(D),(J)(a) of the second list), T. Bartoszynski ((P),(S),(U) of the first list).

We wonder: is the division to “First/Second list” good? Should we add a third middle category, “basic forcing”?

§ 3(A). Desirable Properties: First List.

In this subsection we list various desirable properties and questions and sometimes give a relevant reference (in this paper) but we do not prove anything (whereas §(3C) on contains proofs):

- (A) (α) the ideal $\text{id}(\mathbb{Q}_\lambda)$ is λ^+ -complete, i.e. closed under union of $\leq \lambda$ sets
- (β) the forcing notion \mathbb{Q}_κ is λ -complete (or at least λ -strategically complete)
- (γ) the Boolean Algebra of λ -Borel subsets of ${}^\kappa 2$ modulo the ideal $\text{id}(\mathbb{Q}_\lambda)$ satisfies the λ^+ -c.c., see 3.12(2), note that modulo $\text{id}(\mathbb{Q}_\lambda)$, \mathbb{Q}_λ is dense in this Boolean Algebra this is (E)
- (δ) the forcing notion \mathbb{Q}_λ is λ -bounding, see 0.5(2), §1, §2
- (B) definability of \mathbb{Q}_λ , i.e. \mathbb{Q}_λ is nicely definable, see the definition by induction in §1 and §2, if λ is weakly compact then \mathbb{Q}_λ is λ -Borel, the ideal is similarly definable, see 3.30
- (C) generalizing “adding (forcing) a Cohen real makes the set of old reals null”, see 3.19
- (D) generalizing “adding (i.e. forcing) a λ -random real makes the old real meagre”, see 3.10
- (E) modulo the null ideal, every λ -Borel set is equal to a union of at most λ sets of the form $\lim_\lambda(p), p \in \mathbb{Q}_\lambda$
- (F) can we define integral? We do not know; may we replace $[0, 1]_\mathbb{R}$ as a set of values by some complete linear order? If we waive linearity does it help?
- (G) modulo the ideal every λ -Borel function can be approximated by “steps function of level α ” for many (so unboundedly) many $\alpha < \lambda$ with “step function” being interpreted as $f(\eta) \restriction \alpha$ determined by $\eta \restriction \alpha$ for $\eta \in {}^\lambda 2$, see 3.14
- (H) Lebesgue density theorem, see 3.16, (it means: if the λ -Borel set $\mathbf{B} \subseteq {}^\lambda 2$ is $\text{id}(\mathbb{Q}_\kappa)$ -positive, then for some $\mathbf{B}_1 \in \text{id}(\mathbb{Q}_\lambda)$ for every $\eta \in \mathbf{B} \setminus \mathbf{B}_1$ for some $\alpha < \lambda$ we have $({}^\lambda 2)^{[\eta \restriction \alpha]} \setminus \mathbf{B} \in \text{id}(\mathbb{Q}_\lambda)$);
- (I) Fubini theorem, symmetry, see 3.34

- (J) translation invariance, see 3.24(1)
- (K) permutation invariant, (i.e. for permutations of λ): this works for the \mathbb{Q}_λ'' version, i.e. in §2, see 3.24(2)
- (L) generalizing “if A is a Borel subset of $[0, 1]_\mathbb{R} \times [0, 1]_\mathbb{R}$ of positive measure then A contains a perfect rectangle (even half square)”. But what is perfect? not a copy of ${}^\lambda 2$ but λ -closed set, i.e. the λ -limit of a λ -Kurepa tree, even one with “little pruning in limit levels”
- (M) generalize the random algebra on ${}^\chi 2$ for χ possibly $> \lambda$, see §2, (2.4)
- (N) generalize “modulo the null ideal every Borel set is equal to a union of $\leq \lambda$ sets each λ -closed” and (E) above, see 3.12
- (O) generalize “the sets of reals in a union of a null set and a meagre set”, see 3.17
- (P) Generalize Erdős-Sierpinski theorem: if $2^\lambda = \lambda^+$ or suitable cardinal invariants are equal to λ^+ then there is a permutation of ${}^\lambda 2$ interchanging the null and meagre ideal
- (Q) generalize Borel conjecture: though not connected to random. Now consider:
 - (α) the equivalence of the $\langle \varepsilon_n : n \rangle$ and translated away from meagre set
 - (β) the Σ_n ’s version has an obvious generalization
 - (γ) try shooting through a normal ultrafilter
- (R) dual Borel conjecture, see part II: now the question is:
 - (*) we are given an old set X of λ -reals of cardinality λ^+ , say $X = \{\nu_\alpha : \alpha < \lambda^+\}$. View Cohen_λ as adding a λ -null set: e.g. for $\bar{p} = \langle p_\eta : \eta \in {}^\kappa > 2 \rangle$, $p_\eta \in \mathbb{Q}_\kappa$, $\text{tr}(p_\eta) = \eta$, and clearly p_η is nowhere a cone, but we shall need more
- (S) (selections) Every Σ_1^1 -relation have a contradicting choice function on a positive closed set even in any positive Borel set
- (T) Banach-Tarski paradox may fail for R, R^2 , do it for ${}^\lambda 2$, i.e. there are many
- (U) if generalized “for every meagre A there is a meagre B such that: every $\leq \lambda$ translates of A can be covered by one translation of B ”, but fail for null even for \mathbb{Z} . Generalize to λ .

On raising further problems see [Sh:F1199], concern characters, differentiability, monotonicity (of functions) and going back to the case $\lambda = \aleph_0$.

We have not looked at clauses (P)-(U).

§ 3(B). Desirable Properties. : Second List

Next we consider generalizing results more set theoretic in nature related to forcing (maybe (B)(b),(c),(d) should be here; (A) is treated here, on the others see part II, if at all)

- (A) Chichon’s Diagram

This diagram sums up the provable inequalities between the basic cardinal invariants of the null ideal, the meagre ideal, \mathfrak{d} (the dominating number) and \mathfrak{b} (the undominating number). The basic cardinal invariants of an ideal are the covering number, the additivity number, the cofinality and the non of the ideal, see 0.9.

The diagram gives the provable inequalities among any two invariants (and two equalities each on three invariants). Moreover, under $2^{\aleph_0} \leq \aleph_2$ there are no more connections. Here we fully generalize the ZFC part (for λ inaccessible limit of inaccessibles), see 3.13 and quotations there.

The complementary consistency results, about inequalities of any pair, we intend to deal with in part II.

(B) Generalizing the amoeba forcing

(The amoeba forcing, the one adding a measure zero set including all the old ones, the condition are closed subsets of $[0, 1]_{\mathbb{R}}$ of measure $> \frac{1}{2}$.)

This is natural as the amoeba forcing has been important in set theory of the reals and is closely related to measure, see 3.25 - 3.29.

(C) consistency of “every $A \in \mathcal{P}(\mathbb{R})^{\mathbb{L}[\mathbb{R}]}$ is Lebesgue measurable” (from $\chi > \lambda$ inaccessible).

The problem is: we have names η of λ -reals such that $\text{Levy}(\lambda, < \chi)/\eta$ is not $\text{Levy}(\lambda, < \chi)$.

This certainly occurs for λ -Cohen reals and probably for any other; that is we may add a λ -Cohen $\eta \in {}^\lambda 2$ compare it with \mathbb{Q}_λ shooting a club through $\eta^{-1}\{\ell\}$.

A possible avenue is to consider only “nice $\text{Levy}(\lambda, < \chi)$ -names”, i.e. such that the quotient is $\text{Levy}(\lambda, < \chi)$. In this case there is a “positive” set of λ -reals such that for subsets of it our aim is achieved. We can even define this set of reals. The question is whether this is a “reasonable” or a “forced” solution?

Alternatively we may replace λ -Cohen by another forcing (or ideal) and/or change the collapse; in particular should check the failure for \mathbb{Q}_λ . We also may change the notion of a λ -real, e.g. replace it by $A/(\text{the non-stationary ideal})$ or by filter generated by $\leq \lambda$ subsets of λ ! All this is delayed to part II. We should also check what occurs to sweetness in our present case (see an up-to-date treatment in [RoSh:856]).

We may consider $\{\eta \in {}^\lambda 2 : \eta \text{ is } (\mathbb{Q}, \eta)\text{-generic over } \mathbf{V}_0 \text{ such that every subset of } \lambda \text{ in } \mathbf{V}_0[\eta] \text{ is stationary in } \mathbf{V}\}$, or more. A related question is a complexity of maximal antichains, see 3.31, maybe use measurable cardinals

(D) can we characterize Cohen_λ and \mathbb{Q}_κ among (nicely definable) λ -Borel ideals? Recall Solecki-Kechri’s characterization to Cohen and random (or the ideals) have not looked at it; there are limitations even for $\lambda = \aleph_0$ for [RoSh:628].

(E) In [Sh:480] we show that: for any Suslin c.c.c. forcing, if it adds an undominated real, it adds a Cohen real.

Subsequently some works show relatives, (for other properties), see [Sh:711], [Sh:723].

By [Sh:630], the only necessary “nice” c.c.c. forcing commuting with Cohen is Cohen itself. Do we have a parallel?

- (F) We know much on ultrafilters on \mathbb{N} . Also we have considerable knowledge about λ -complete ultrafilters on λ or higher cardinals when λ is a measurable cardinal. Not much set theoretic work was done on regular ultrafilter, but see in recent years on reasonable ultrafilter in [Sh:830], Roslanowski-Shelah [RoSh:860], [RoSh:890] and recently on ultrafilters related to Keisler order Malliaris-Shelah [MiSh:996].

See [BnSh:642]; for an ultrafilter D on λ recall that $\chi(D)$ is the character, $\pi\chi(D)$ pseudo-character (= minimal cardinality of $\mathcal{A} \subseteq [\lambda]^\lambda$ such that $(\forall B \in D)(\exists A \in \mathcal{A})[A \subseteq B]$, note that $A \in [\lambda]^\lambda$ is not necessarily in D ! As in [RoSh:860] dealing with reasonable ultrafilters we may consider the Borel version (i.e. the minimal number of Borel subsets of D which generate it) and λ -real version. Then as in “reasonable ultrafilter” we can show $\text{CON}(\text{for every uniform ultrafilter } D \text{ on } \lambda, \pi\chi_{2\text{-real}}(D) = \lambda^+ < 2^\lambda)$.

What about the ultrafilter forcing: reasonable ultrafilter on λ can be generated by $< 2^\kappa$ sets? can force a creature condition diagonalizing a uniform ultrafilter on λ .

- (G) Related is Galvin-Prikry theorem which says that for a Borel (or even Σ_1^1) subset $\mathcal{P}(N/\mathbb{N})$ for some set $A \in [\mathbb{N}]^{\aleph_0}$, the set $[A]^{\aleph_0}$ is included in or disjoint from \mathbf{B} .
- (H) consistency of Moore conjecture, X is λ -first countable (analog of first countable). Of course we can prove it using Dow lemma which holds for adding many λ -Cohens, so not clear how interesting
- (I) preserving “ η is \mathbb{Q}_κ -generic” parallel to [Sh:c, Ch.XVIII,§3], [Sh:c, Ch.VI,§3]
- (J) (a) connected $\text{cf}(\mathbb{Q}_\kappa)$ and Chichon’s diagram and number of reasonable generators of an ultrafilter
- (b) just covering a λ -Cohen covering of \mathbb{Q}_κ are lower bounds; reason:
 $\eta \restriction \alpha \cup \langle 1 - \eta(\beta) : \beta \in [\alpha, \kappa) \rangle$ is generic.

§ 3(C). On \mathbb{Q}_κ .

A general frame including 3.1 is:

Definition 3.4. 1) Let $\text{id}(\text{Cohen}_\kappa)$ the set of κ -meagre subsets of ${}^\kappa 2$, i.e. $\{A \subseteq {}^\kappa 2 : A \subseteq \cup \{\lim_\kappa(\mathcal{T}_i) \text{ for } i < i(*)\}\}$ where $i(*) \leq \kappa$ each \mathcal{T}_i a nowhere dense subtree of ${}^\kappa 2$, i.e. $({}^{\kappa>2}, \trianglelefteq)$.

2) We say \mathbf{i} is an ideal case when \mathbf{i} consists of (letting $\mathbb{Q} = \mathbb{Q}_\mathbf{i}$, etc.):

- (a) κ is a cardinal (here usually regular)
- (b) \mathbb{Q} is a forcing notion
- (c) η is a \mathbb{Q} -name of a member of ${}^\kappa 2$
- (d) (α) each $p \in \mathbb{Q}$ is a subtree of $({}^{\kappa>2}, \trianglelefteq)$ and let $\mathbf{B}_p = \mathbf{B}_{\mathbf{i},p} = \lim_\kappa(p)$
or at least we have
- (β) $p \mapsto B_p = \mathbf{B}_{\mathbf{i},p} \subseteq {}^\kappa 2$ a κ -Borel subset of ${}^\kappa 2$ decreasing with p such that $p \Vdash \text{“}\eta \in \mathbf{B}_p\text{”}$; so really the function $p \mapsto \mathbf{B}_p$ is part of \mathbf{i}

3) For $\mathbf{i} = (\kappa, \mathbb{Q}, \eta)$ let $\text{id}_1^1 = \text{id}_1(\mathbf{i})$ be $\{A \subseteq {}^\kappa 2 : \text{for some } \lambda\text{-Borel set } \mathbf{B} \text{ we have } A \subseteq \mathbf{B} \text{ and } \Vdash_{\mathbb{Q}} \text{"}\eta \notin \mathbf{B}\text{"}\}$; we may omit the 1.

4) For $\mathbf{i} = (\kappa, \mathbb{Q}, \eta)$ let $\text{id}_1^2 = \text{id}_2(\mathbf{i})$ be the following ideal on ${}^\kappa 2 : A \in \text{id}_1^2$ iff:

(a) $A \subseteq {}^\kappa 2$

(b) there are $i(*) \leq \kappa$ and pre-dense subsets \mathcal{I}_i of \mathbb{Q} for $i < i(*)$ such that $A \subseteq \{\eta \in {}^\kappa 2 : \text{if } i < i(*) \text{ then } \eta \text{ fulfills } \mathcal{I}_i\}$, see below.

5) We say $\eta \in {}^\kappa 2$ fulfill \mathcal{I} , a subset of \mathbb{Q} when $(\exists p \in \mathcal{I})(\eta \in \mathbf{B}_p)$.

Remark 3.5. For \mathbf{i} an ideal case do we have $\text{id}_1(\mathbf{i}) = \text{id}_2(\mathbf{i})$, if forcing with $\mathbb{Q}_{\mathbf{i}}$ satisfies “every new $X \in [\text{Ord}]^{\leq \kappa}$ is included in some old $Y \in [\text{Ord}]^{\leq \kappa}$ ”? Delayed to part II.

Definition 3.6. 1) For \mathbf{i} as in 3.4, we define $\text{cov}(\mathbf{i}), \text{add}(\mathbf{i}), \text{non}(\mathbf{i}), \text{cf}(\mathbf{i})$ as those numbers for the ideal id_1 , see 0.9.

2) If $\kappa_{\mathbf{i}}, \eta_{\mathbf{i}}$ are clear from $\mathbb{Q}_{\mathbf{i}}$ we may write \mathbb{Q} instead of \mathbf{i} ; in particular for \mathbb{Q}_{κ} from §1 and for Cohen_{κ} .

Recalling $S_{\text{inc}}^{\kappa} = \{\partial : \partial < \kappa \text{ is inaccessible}\}$, note that:

Claim 3.7. *If $\kappa > \sup(S_{\text{inc}}^{\kappa})$ then for some open dense subsets $\mathcal{I}_1, \mathcal{I}_2$ of \mathbb{Q}_{κ} , Cohen_{κ} respectively, we have $\mathbb{Q}_{\kappa} \restriction \mathcal{I}_1 \cong \text{Cohen}_{\kappa} \restriction \mathcal{I}_2$.*

Proof. Let $\mu = \sup(S_{\text{inc}}^{\kappa})$.

Let $\mathcal{I}_1 = \{p \in \mathbb{Q}_{\kappa} : \ell g(\text{tr}(p)) \geq \mu\}$, $\mathcal{I}_2 = \{\eta \in \text{Cohen}_{\kappa} : \ell g(\eta) \geq \mu\}$ and $F : \mathcal{I}_1 \rightarrow \mathcal{I}_2$ be $F(p) = \text{tr}(p)$. □_{3.7}

Claim 3.8. 1) $\text{id}(\mathbb{Q}_{\kappa})$ is a κ^+ -complete ideal on ${}^\kappa 2$ and also $\text{id}(\text{Cohen}_{\kappa})$ is.

2) If κ is weakly compact and $\mathcal{I}_{\alpha} \subseteq \mathbb{Q}_{\kappa}$ is pre-dense for $\alpha < \alpha_* < \kappa^+$ then the sets $\mathcal{I}_1, \mathcal{I}_2$ are dense open subsets of \mathbb{Q}_{κ} where

$$\mathcal{I}_1 = \{p \in \mathbb{Q}_{\kappa} : \text{if } \eta \in \lim_{\kappa}(p) \text{ and } \alpha < \alpha_* \text{ then } \eta \text{ fulfills } \mathcal{I}_{\alpha}, \text{ i.e. } (\exists q \in \mathcal{I}_{\alpha})(\eta \in \lim_{\kappa}(q))\}$$

$$\mathcal{I}_2 = \{p \in \mathbb{Q}_{\kappa} : \text{for every } \alpha < \alpha_* \text{ there is } \partial < \kappa \text{ such that } [\eta \in p \cap {}^\partial 2 \Rightarrow p^{[\eta]} \text{ is above some } q \in \mathcal{I}_{\alpha}]\}.$$

Proof. See the proof of 2.24. □_{3.8}

Observation 3.9. 1) If $p, q \in \mathbb{Q}_{\kappa}$ and $\mathbb{Q}_{\kappa} \models \text{"}p \not\leq q\text{"}$ then for some r , we have $q \leq_{\mathbb{Q}_{\kappa}} r$ and r, p are incompatible (so $\lim_{\kappa}(p), \lim_{\kappa}(r)$ are disjoint).

2) If $p_1, p_2 \in \mathbb{Q}_{\kappa}$ then the following conditions are equivalent:

- (a) p_1, p_2 are compatible
- (b) $\lim_{\kappa}(p_1) \cap \lim_{\kappa}(p_2)$ are not disjoint
- (c) $\text{tr}(p_1) \in p_2$ and $\text{tr}(p_2) \in p_1$
- (d) $\text{tr}(p_1) \trianglelefteq \text{tr}(p_2) \in p_1$ or $\text{tr}(p_2) \trianglelefteq \text{tr}(p_1) \in p_2$.

Proof. 1) By (2).

2) First, (a) \Rightarrow (b) as $r \in \mathbb{Q}_{\kappa} \Rightarrow \lim_{\kappa}(r) \neq \emptyset$ by 1.6(1).

Second, (b) \Rightarrow (c) as $\eta \in \lim_{\kappa}(c) \not\Rightarrow \text{tr}(r) \trianglelefteq \eta$.

Third, (c) \Rightarrow (d) trivially.

Fourth, $(d) \Rightarrow (a)$ as without loss of generality $\text{tr}(p_1) \leq \text{tr}(p_2) \in p_1$, hence $p_1^{[\text{tr}(p_2)]}, p_2$ are members of \mathbb{Q}_κ with the same trunk so are compatible by 1.5. As $\mathbb{Q}_\kappa \models "p_2 \leq p_1^{[\text{tr}(p_2)]}"$, we are done. $\square_{3.9}$

Claim 3.10. 1) If κ is an inaccessible limit of inaccessibles, then in $\mathbf{V}^{\mathbb{Q}_\kappa}$ the set $(\kappa 2)^{\mathbf{V}}$ is κ -meagre.
2) For κ as above $\text{cov}(\mathbb{Q}_\kappa) \leq \text{non}(\text{Cohen}_\kappa)$.

Remark 3.11. The dual is 3.19.

Proof. Let $\langle \partial_i : i < \kappa \rangle$ list in increasing order the (strongly) inaccessible cardinals below κ .

We claim

$$\Vdash_{\theta_\kappa} \text{ " if } \nu \in (\kappa 2)^{\mathbf{V}} \text{ then for every } i < \kappa \text{ large enough } \eta \restriction (\partial_i + 1, \partial_{i+1}) \not\subseteq \nu \text{ " .}$$

This clearly suffices. Let $p \in \mathbb{Q}_\kappa$ and we shall fix $\nu \in (\kappa 2)^{\mathbf{V}}$ and we shall find α, q and $i_* < \kappa$ such that $p \leq_{\mathbb{Q}_\kappa} q$ and $q \Vdash$ "if $i > i_*$ then $\eta \restriction (\partial_i + 1, \partial_{i+1}) \not\subseteq \nu$ ".

Let i_* be such that $\ell g(\text{tr}(p)) < \partial_{i_*}$ and let $(\varrho, S_1, \bar{\Lambda})$ be a witness for $p \in \mathbb{Q}_\kappa$. Now let $S_2 = \{\partial_{i+1} : i > i_*\}$ and if $\partial = \partial_{i+1} \in S_2$ let $\mathcal{J}_\partial = \{\nu \in {}^\partial 2 : \eta \restriction (\partial_i + 1, \partial_{i+1}) \not\subseteq \nu\}$, clearly a dense open subset of \mathbb{Q}_∂ . Now let $S' = S_1 \cup S_2$ and for $\partial \in S'$ let Λ'_∂ be Λ_∂ if $\partial \in S_1 \setminus S_2$ and $\Lambda_\partial \cup \{\mathcal{J}_\partial\}$ if $\partial \in S_1 \cap S_2$ and $\{\mathcal{J}_\partial\}$ if $\partial \in S_2 \setminus S_1$, lastly $\bar{\Lambda}' = \langle \Lambda'_\partial : \partial \in S' \rangle$.

Easily $(\text{tr}(p), S', \bar{\Lambda}')$ witness some $q \in \mathbb{Q}_\kappa$ which is as required. $\square_{3.10}$

Claim 3.12. 1) $[\kappa \text{ weakly compact}]$ Any κ -Borel set \mathbf{B} is equal modulo $\text{id}(\mathbb{Q}_\kappa)$ to the union of $\leq \kappa$ sets each is κ -closed and even \mathbb{Q}_κ -basic, see Definition 0.2(2); so $\text{id}_1(\mathbb{Q}_\kappa) = \text{id}_2(\mathbb{Q}_\kappa)$.

2) $\text{Borel}_\kappa / \text{id}_2(\mathbb{Q}_\kappa)$ is a κ^+ -c.c. Boolean Algebra.

Proof. 1) As \mathbb{Q}_κ satisfies the κ^+ -c.c. it is enough to show that for a dense set of $p \in \mathbb{Q}_\kappa$, $\lim_\kappa(p_\kappa)$ is $\subseteq \mathbf{B}$ or disjoint to \mathbf{B} , this easily holds by 3.8(2).

This implicitly uses " \mathbb{Q}_κ is κ -bounding".

2) Should be clear. $\square_{3.12}$

§ 3(D). Chichon's Diagram.

Claim 3.13. The parallel of all inequalities in Chichon's diagram holds. That is the inequalities implicitly in the diagram and

$$\begin{aligned} \text{add}(\text{id}(\text{Cohen}_\lambda)) &= \min\{\text{cov}(\text{id}(\text{Cohen}_\lambda)), \mathfrak{b}_\lambda\} \\ \text{cf}(\text{id}(\text{Cohen}_\lambda)) &= \max\{\text{non}(\text{Cohen}_\lambda), \mathfrak{d}_\lambda\}. \end{aligned}$$

Proof. For any ideal \mathbb{I} recalling Definition 0.9 we have $\text{add}(\mathbb{I}) \leq \text{non}(\mathbb{I}) \leq \text{cf}(\mathbb{I})$ and $\text{add}(\mathbb{I}) \leq \text{cov}(\mathbb{I}) \leq \text{cf}(\mathbb{I})$ and in our case, as both ideals are λ^+ -complete clearly all values are in the interval $[\lambda^+, 2^\lambda)$. Also $\text{cov}(\mathbb{Q}_\lambda) \leq \text{add}(\text{Cohen}_\lambda)$ by 3.10, so recalling 3.21 the inequalities (and equalities) involving $\mathfrak{b}_\lambda, \mathfrak{d}_\lambda$ holds.

Now $\text{add}(\mathbb{Q}_\lambda) \leq \text{add}_\lambda(\text{Cohen}_\lambda)$ by 3.22. Also $\text{add}(\text{Cohen}_\lambda) \leq \text{add}(\mathbb{Q}_\lambda)$ by 3.22 and $\text{cf}(\text{Cohen}_\lambda) \leq \text{cf}(\mathbb{Q}_\lambda)$ by 3.23(1) and $\text{cov}(\text{Cohen}_\lambda) \leq \text{non}(\mathbb{Q}_\lambda)$ by 3.23(2) and $\text{cov}(\mathbb{Q}_\kappa) \leq \text{non}(\text{Cohen}_\lambda)$ by 3.23(3) so we are done. $\square_{3.13}$

Claim 3.14. [κ weakly compact] Assume F is a κ -Borel function from ${}^\kappa 2$ to ${}^\kappa 2$. For a dense set of p 's, F can be read continuously on $\lim_\kappa(p)$, i.e. for some club E of κ and $\bar{h} = \langle h_\alpha : \alpha \in C \rangle$ we have:

- $h_\alpha : p \cap {}^\alpha 2 \rightarrow {}^\alpha 2$
- if $\eta \in p \cap {}^\alpha 2, \nu \in p \cap {}^\beta 2, \eta \triangleleft \nu$ and $\{\alpha, \beta\} \subseteq C$ then $h_\alpha(\eta) \triangleleft h_\beta(\nu)$
- if $\eta \in \lim_\kappa(p)$ then $F(\eta) = \cup \{h_\alpha(\eta \upharpoonright \alpha) : \alpha \in C\}$.

Remark 3.15. This is parallel to “every Borel function $F : [0, 1] \rightarrow [0, 1]$ can be approximated by step functions, that is functions such that for some finite partitions of $[0, 1]$ to intervals, it is constant on each interval.

Proof. Clearly it suffices to show that for some unbounded subset C of κ because then its closure is as required. Now use 2.24 for the name $F(\eta) \upharpoonright \alpha$ for $\alpha < \kappa$, i.e. $\mathcal{I}_\alpha = \{p : p \text{ forces a value to } F(\eta) \upharpoonright \alpha\}$. \square

Concerning Lebesgue Density Theorem:

Conclusion 3.16. [κ weakly compact] If $X \subseteq {}^\kappa 2$ is κ -Borel, then for some $Y \in \text{id}(\mathbb{Q}_\kappa)$ for every $\eta \in X \setminus Y$ for every $\alpha < \kappa$ large enough $(2^\kappa)^{[\eta \upharpoonright \alpha]} \setminus X$ belongs to the ideal.

Claim 3.17. If λ is limit of inaccessibles then ${}^\lambda 2$ can be partitioned to two sets A_0, A_1 such that A_0 is in $\text{id}(\text{Cohen}_\lambda)$ and A_1 is in $\text{id}(\mathbb{Q}_\lambda)$.

Proof. Let $\langle \kappa_i : i < \lambda \rangle$ list the inaccessibles $< \lambda$ in increasing order. For $\eta \in {}^\lambda 2$ let $S_\eta = \{\kappa_{i+1} : i < \lambda \text{ and } \kappa_i \geq \ell g(\eta)\}$ and for $\kappa_{i+1} \in S_\eta$ let $\mathcal{I}_{\kappa_{i+1}} = \{q \in \mathbb{Q}_{\kappa_i} : \ell g(\text{tr}(q)) > \kappa_i \text{ and } \text{tr}(q) \upharpoonright [\kappa_i, \ell g(q)) \text{ is not constantly zero}\}$.

Lastly, let $p_\eta \in \mathbb{Q}_\lambda$ be witnessed by $(\varrho, \{\kappa_{i+1} : \kappa_i > \ell g(\eta)\}, \langle \mathcal{I}_{\kappa_{i+1}} : i < \lambda \text{ and } \kappa_i \geq \ell g(\eta) \rangle)$.

Now

- ⊞ (a) p_η indeed belongs to \mathbb{Q}_λ
- (b) $\text{tr}(p_\eta) = \eta$
- (c) p_η is a no-where dense subtree of ${}^\lambda 2$.

Let $A_0 = \cup \{\lim(p_\eta) : \eta \in {}^\lambda 2\}, A_1 = {}^\lambda 2 \setminus A_0$, now check.

Concerning Chichon's diagram, it seems the situation here is different. $\square_{3.17}$

Definition 3.18. 1) Let $\mathbf{nwst}_\lambda = \mathbf{nwst}_\lambda(S_{\text{inac}}^\lambda)$, see below.

2) For $S_* \subseteq \lambda$ unbounded let $\mathbf{nwst}_\lambda(S_*) = \min\{|\mathcal{S}| : \mathcal{S} \text{ is a set of nowhere stationary subsets of } S_* \text{ such that we cannot find a nowhere stationary cover } \langle S_\alpha : \alpha < \lambda \rangle \text{ of } \mathcal{S}, \text{ see below}\}$.

3) We say \bar{S} is a nowhere stationary S_* -cover of \mathcal{S} when ($S_* \subseteq \lambda = \sup(S_*)$ and):

- (a) \mathcal{S} a set of nowhere stationary subsets of S_*
- (b) $\bar{S} = \langle S_\alpha : \alpha < \lambda \rangle$
- (c) each S_α is nowhere stationary $\subseteq S_*$
- (d) $(\forall S \in \mathcal{S})(\exists \alpha < \lambda)(S \subseteq S_\alpha \text{ mod } J_\lambda^{\text{bd}})$.

We intend to deal with it in the second part.

Claim 3.19. *If κ is inaccessible limit of inaccessibles and \mathbf{V}_1 is an extension of \mathbf{V} (e.g. a forcing extension) then $\mathbf{V}_1 \models “(\kappa^2)^\mathbf{V} \in \text{id}(\mathbb{Q}_\kappa)”$ provided that at least one of the following holds:*

- (a) $\mathbf{V}_1 = \mathbf{V}^{\text{Cohen}(\kappa)}$, see Definition 0.5(2)
- (b) in \mathbf{V}_1, κ is still inaccessible and there are sequences $\bar{\eta} = \langle \eta_\partial : \partial \in S \rangle, \bar{\alpha} = \langle \alpha_\partial : \partial \in S \rangle$ such that
 - (α) $S \subseteq \kappa$
 - (β) $\partial \in S \Rightarrow \alpha_\gamma = \sup(S \cap \gamma) < \gamma$
 - (γ) S is a set of inaccessibles (in \mathbf{V}_1 hence in \mathbf{V})
 - (δ) $\eta_\partial \in {}^\partial 2$
 - (ε) if $\eta \in (\kappa^2)^\mathbf{V}$ then for unboundedly many $\partial \in S$ we have $\eta(\alpha_\gamma, \gamma) \subseteq \eta_\partial$
- (c) in \mathbf{V}_1, κ is still inaccessible limit of inaccessibles but $\mathcal{H}(\kappa)^\mathbf{V} \neq \mathcal{H}(\kappa)^{\mathbf{V}_1}$
- (d) like clause (b) but
 - (β)' S is nowhere stationary
 - (δ)' $\bar{\Lambda} = \langle \Lambda_\partial : \partial \in S \rangle, \Lambda_\partial$ a set $\leq \partial$ dense subset of \mathbb{Q}_∂
 - (ε)' if $\eta \in (\kappa^2)^\mathbf{V}$ then for unboundedly many $\partial \in S, \eta \restriction \partial$ does not fulfill Λ_∂ .

Proof. Case (a):

It suffices to prove that the assumptions of (b) holds. Clearly the forcing pre-serves inaccessibility, let $\eta \in {}^\kappa 2$ be the name of the λ -Cohen real and let:

- $S_1 = \{\partial < \kappa : \partial \text{ inaccessible}\}$
- $S = \{\partial \in S_1 : \partial > \sup(S_1 \cap \partial)\}$
- $\eta_\partial = \eta \restriction \partial$

Clearly clauses (α), (γ) are satisfied by S_1 and by S and clause (β) is satisfied by S and for any $\eta \in {}^\kappa 2$, the derived sequence $\langle \eta_\partial : \partial \in S \rangle$ satisfies clause (δ) by our choice above.

Lastly, clause (ε) holds as $\text{Cohen}_\kappa = (\kappa^{>2}, \triangleleft)$, so the assumptions of clause (b) holds.

Clause (b):

For $\partial \in S$ let $\mathcal{I}_\partial^* = \{q \in \mathbb{Q}_\partial : \text{if } \nu \in {}^{(\alpha_\partial)} 2 \text{ then } \nu^\wedge(\eta_\partial \restriction [\alpha_\partial, \partial)) \notin \lim_\partial(q)\}$, clearly dense open subset of \mathbb{Q}_κ . Now in \mathbf{V} , let $\mathcal{I} = \{p \in \mathbb{Q}_\kappa : \text{for } (\varrho_p, S_p, \bar{\Lambda}_p) \text{ witnessing } p \in \mathbb{Q}_\kappa \text{ we have } \partial \in S \setminus \ell g(\varrho_p) \Rightarrow \partial \in S_p \wedge \mathcal{I}_p \in \Lambda_{p, \partial}\}$.

Clause (c):

Let S_1 be the set of inaccessibles in \mathbf{V}_1 which are $< \kappa$. Let $\alpha < \kappa$ and ν be such that $\nu \in (\alpha^2)^{\mathbf{V}_1}$ but $\nu \notin (\alpha^2)^\mathbf{V}$.

Now let

- $S_1 = \{\partial \in S_1 : \partial > \alpha \text{ and } \partial > \sup(S \cap \partial)\}$
- $\mathcal{I} = \{p \in \mathbb{Q}_\partial : \text{for some } \beta \text{ we have } \beta + \alpha \leq \ell g(\text{tr}(p)) \text{ and } \langle \text{tr}(p)(\beta + i : i < \alpha) \rangle = \nu\} \text{ for } \partial \in S$
- $\Lambda_\partial = \{\mathcal{I}_\partial\}$ for $\partial \in S$.

Easy, the assumptions of clause (d) holds, so the results follow.

Clause (d): Like the proof of clause (b).

□_{3.19}

Observation 3.20. *If $X \subseteq {}^\kappa 2$ is meagre and $A \subseteq \kappa$ is unbounded then there is an increasing sequence $\bar{\alpha}$ of member of A of length κ and $\eta \in {}^\kappa 2$ such that $X \subseteq X_{\eta, \bar{\alpha}}$ where*

- $X_{\eta, \bar{\alpha}} = \{\nu \in {}^\kappa 2: \text{for every } i < \kappa \text{ large enough, } \eta \upharpoonright [\alpha_i, \alpha_{i+1}) \not\subseteq \nu\}$ hence
- $i < \kappa \Rightarrow X_{\eta, \langle \alpha_{i+j}: j < \kappa \rangle} = X_{\eta, \bar{\alpha}}$.

Proof. As in earlier cases. □_{3.20}

Observation 3.21. 1) $\text{add}(\text{Cohen}_\kappa) \leq \mathfrak{b}_\kappa \leq \text{non}(\text{Cohen}_\kappa)$.

2) $\text{cov}(\text{Cohen}_\kappa) \leq \mathfrak{d}_\kappa \leq \text{cf}(\text{Cohen}_\kappa)$.

3) $\text{add}(\text{Cohen}_\kappa) = \min\{\mathfrak{b}_\kappa, \text{cov}(\text{Cohen}_\kappa)\}$.

4) $\text{cf}(\text{Cohen}_\kappa) = \min\{\mathfrak{d}_\kappa, (\text{cf}(\text{Cohen}_\kappa))\}$.

Proof. As for $\kappa = \aleph_0$, part is given in details in the proof of 3.22. □_{3.21}

Claim 3.22. $\text{add}(\mathbb{Q}_\kappa) \leq \text{add}(\text{Cohen}_\kappa)$.

Proof. By 3.7, without loss of generality $\kappa = \sup(S_{\text{inc}}^\kappa)$.

So let $\mu = \text{add}(\text{Cohen}_\kappa)$ and $\langle X_\zeta : \zeta < \mu \rangle$ be a sequence of κ -meagre sets with no κ -meagre set including all of them. Let $A = S_{\text{inc}}^\kappa$ and applying 3.20 to κ, X_ζ, A we get η_ζ, S_ζ , we write $\bar{\partial}_\zeta$ and not $\bar{\alpha}_\zeta$ as it is a sequence of inc.

For each ζ , let

- $S_\zeta = \{\bar{\partial}_{\zeta, i+1} : i < \kappa\}$ and for $\partial \in S_\zeta$ let
- $\Lambda_{\zeta, \partial} = \{\mathcal{I}_{\zeta, \alpha} : \alpha < \partial\}$ where
- $\mathcal{I}_{\zeta, \partial} = \{p \in \mathbb{Q}_\partial : \ell g(\text{tr}(p)) > \alpha \text{ and } \text{tr}(p) \upharpoonright [\alpha, \ell g(\text{tr}(p))) \not\subseteq \eta_\zeta\}$
- $\mathcal{I}_{\zeta, \alpha} = \{p \in \mathbb{Q}_\kappa : \text{letting } (tr(p), S_p, \bar{\Lambda}_p) \text{ witness } p \text{ if } \partial > \alpha \text{ satisfies } \ell g(\text{tr}(p)) < \partial \in S_\zeta \text{ then } \partial \in S_p \text{ and } \Lambda_{\zeta, \partial} \subseteq \Lambda_{p, \partial}\}$.

Now

- (*) (a) $\Lambda_{\zeta, \partial}$ is a set of $\leq \partial$ dense open subsets of \mathbb{Q}_∂ for $\partial \in S_{\text{inc}}^\kappa, \zeta < \mu$
- (b) \mathcal{I}_ζ is a dense open subset of \mathbb{Q}_κ for $\zeta < \mu$.

Now clearly it suffices to prove that

- ⊞ there is no sequence $\langle \mathcal{I}_\varepsilon : \varepsilon < \kappa \rangle$ of dense open subsets of \mathbb{Q}_κ such that $\zeta < \mu \Rightarrow \text{set}(\mathcal{I}_\zeta) \subseteq \bigcup_{\varepsilon < \kappa} \text{set}(\mathcal{I}_\varepsilon)$, see Definition xxx.

Why ⊞ holds? Let $\langle \mathcal{I}_\varepsilon : \varepsilon < \kappa \rangle$ be a counterexample.

Let $\Omega_\varepsilon = \{\text{tr}(p) : p \in \mathcal{I}_\varepsilon\}$ and choose $\bar{p}_\varepsilon = \langle p_{\varepsilon, \rho} : \rho \in \Omega_\varepsilon \rangle$ be such that $p_{\varepsilon, \rho} \in \mathcal{I}_\rho$ so

- (*) \bar{p}_ε is predense and $\cup\{\lim_\kappa(p_{\varepsilon, \rho}) : \rho \in \Omega_\varepsilon\} \supseteq \text{set}(\mathcal{I}_\varepsilon)$
- (*) let $(\text{tr}(p_{\varepsilon, \rho}), S_{\varepsilon, \rho}, \bar{\Lambda}_{\varepsilon, \rho})$ witness $p_{\varepsilon, \rho} \in \mathbb{Q}_\kappa$.

Now

- (*) let $S = \{\partial \in S_{\text{inc}}^\kappa : \text{for some } \varepsilon < \partial \text{ and } \rho \in \Omega_\varepsilon \cap {}^\kappa > 2 \text{ we have } \partial \in S_{\varepsilon, \rho}\}$
- (*) for $\partial \in S$ let $\Lambda_\partial = \cup\{\Lambda_{\varepsilon, \rho} : \varepsilon < \partial \text{ and } \partial \in \Omega_\varepsilon \cap {}^\kappa > 2\}$.

The rest is close to 3.17.

Now S is nowhere stationary hence there is an increasing continuous sequence $\bar{\alpha} = \langle \alpha_i : i < \kappa \rangle$ of ordinals from $\kappa \setminus S_i$ each α_{i+1} inaccessible. By induction on $i \leq \kappa$ we choose $\eta_i \in {}^{\alpha_i}2$ such that

- $j < i \Rightarrow \eta_j \triangleleft \eta_i$
- if $i = j + 1, \rho \in {}^{(\alpha_j)}2$ then $\rho^\wedge(\eta_i \upharpoonright [\alpha_j, \alpha_i)) \in \text{set}(\Lambda_{\alpha_i})$.

Now $X_{\eta_\kappa, \bar{\alpha}}$ is a κ -meagre set including every X_ζ for $\zeta < \mu$, contradiction. $\square_{3.22}$

Claim 3.23. 1) $\text{cf}(\text{Cohen}_\lambda) \leq \text{cf}(\mathbb{Q}_\kappa)$.

2) $\text{cov}(\text{Cohen}_\lambda) \leq \text{non}(\mathbb{Q}_\lambda)$.

3) $\text{cov}(\mathbb{Q}_\lambda) \leq \text{non}(\text{Cohen}_\lambda)$.

Proof. Similarly to 3.22. \square

Claim 3.24. 1) Considering ${}^\kappa 2$ as an Abelian Group (addition in modulo 2, coordinatewise), the ideal $\text{id}(\mathbb{Q}_\kappa)$ is closed under translation, i.e. if $\mathbf{B} \subseteq {}^\lambda 2$ and $\eta \in {}^\lambda 2$ then $\mathbf{B} \in \text{id}(\mathbb{Q}_\lambda) \Leftrightarrow \eta \oplus \mathbf{B} \in \text{id}(\mathbb{Q}_\lambda)$ where $\eta \oplus \mathbf{B} := \{\eta \oplus \nu : \nu \in \mathbf{B}\}$.

2) If $\mathbb{Q}_\kappa = \mathbb{Q}_\kappa$, $\text{id}(\mathbb{Q}_\kappa)$ is mapped onto itself by $\hat{\pi}$ for every permutation π of λ where $\hat{\pi} \in \text{Sym}({}^\lambda 2)$ is defined by $\eta \in {}^\lambda 2 \Rightarrow \hat{\pi}(\eta) = \langle \eta(\pi^{-1}(\alpha)) : \alpha < \lambda \rangle$.

Proof. Straight. $\square_{3.24}$

* * *

What about the parallel to “amoeba forcing”?

Definition 3.25. 1) For $\alpha < \kappa, \nu \in {}^\alpha 2, p \in \mathbb{Q}_\kappa, \eta \in p \cap {}^\alpha 2$ let $p^{[\eta, \nu]} = \{\rho : \rho \leq \nu \text{ or for some } \varrho \text{ we have } \eta^\wedge \varrho \in p \wedge \rho = \nu^\wedge \varrho\}$.

2) For $\mathcal{J} \subseteq \mathbb{Q}_\kappa, \alpha < \kappa$ and permutation π of ${}^\alpha 2$ let $\mathcal{J}^{[\alpha, \pi]} = \{p^{[\eta, \nu]} : p \in \mathcal{J}, \eta \in {}^\alpha 2, \nu = \pi(\eta)\}$.

3) For $\Lambda \subseteq \{\mathcal{J} : \mathcal{J} \subseteq \mathbb{Q}_\kappa\}$

- for $\alpha < \kappa$ and $\pi \in \text{Sym}({}^\alpha 2)$ let $\Lambda^{[\alpha, \pi]} = \{\mathcal{J}^{[\alpha, \pi]} : \mathcal{J} \in \Lambda\}$
- for $\alpha < \kappa$ let $\Lambda^{[\alpha]} = \cup \{\mathcal{J}^{[\alpha, \pi]} : \pi \in \text{Sym}({}^\alpha 2)\}$
- for $\alpha < \kappa$ let $\Lambda^{<\alpha>} = \cup \{\Lambda^{[\beta]} : \beta < \alpha\}$
- let $\lambda^{<\alpha>} = \{\mathcal{J} : \text{for some } \mathcal{J} \in \Lambda \text{ and } \pi, \mathcal{J}^{[\pi, \alpha]} = \mathcal{J}^{[\alpha]}\}$.

4) We say $\Lambda \subseteq \{\mathcal{J} : \mathcal{J} \subseteq \mathbb{Q}_\kappa\}$ is nice when:

- $\Lambda^{<\alpha>} \subseteq \Lambda$ for every $\alpha < \kappa$

(hence if $\mathcal{J}_1 \in \Lambda, \mathcal{J}_2 \subseteq \mathbb{Q}, \alpha < \kappa$ and $\mathcal{J}_1^{[\alpha]} = I_2^{[\alpha]}$ then $I_2 \in \Lambda$).

5) For $p \in \mathbb{Q}_\kappa$ let $\text{nb}(p) = \{p^{[\eta, \nu]} : \eta \in p \cap {}^\alpha 2, \nu \in {}^\alpha 2 \text{ for some } \alpha < \kappa\}$.

Claim 3.26. 1) If $\alpha < \kappa$ and $\mathcal{J} \subseteq \mathbb{Q}_\kappa$ is open/dense/predense then so is $\mathcal{J}^{[\alpha]}$ in \mathbb{Q}_α .

2) If $\Lambda \subseteq \{\mathcal{J} : \mathcal{J} \subseteq \mathbb{Q}_\kappa\}$ then $|\Lambda^{[\alpha]}|, |\Lambda^{<\alpha>}| \leq |\Lambda| + \kappa$ for $\alpha < \kappa$ and $|\Lambda^{<\alpha>}| \leq |\Lambda| + \kappa$ for $\alpha \leq \kappa$.

3) If $\Lambda \subseteq \{\mathcal{J} : \mathcal{J} \subseteq \mathbb{Q}_\kappa \text{ is predense}\}$ has cardinality $\leq \kappa$ then so are $\Lambda^{<\kappa>}, \Lambda^{[\kappa]}$, the latter is nice.

- Claim 3.27.** 1) If $p \in \mathbb{Q}_\kappa$ then $\text{nb}(p)$ is a predense subset of \mathbb{Q}_κ .
 2) If $p \in \mathbb{Q}_\kappa$ then $\text{set}(\text{nb}(p))$ is equal to $\{\eta \in {}^\kappa 2 : \text{there is } \nu \in \text{lim}_\kappa(p) \text{ such that } (\forall^\infty \alpha < \kappa)(\eta(\alpha) = \nu(\alpha))\}$.
 3) If $X \in \text{id}(\mathbb{Q}_\kappa)$ then for some $p \in \mathbb{Q}_\kappa$ such that $\text{set}(\text{nb}(p)) \subseteq X$.

Definition 3.28. Let $\mathbb{Q}_\kappa^{\text{am}}$ be the following forcing notions:

- (A) the member of $\mathbb{Q}_\kappa^{\text{am}}$ has the form $[\alpha, p]$ with $\alpha < \kappa, p \in \mathbb{Q}_\kappa$
- (B) the order on $\mathbb{Q}_\kappa^{\text{am}}$ is: $(\alpha_1, p_1) \in [\alpha_2, p_2]$ iff sn
 - (a) $\alpha_1 \leq \alpha_2$
 - (b) $p_1 \leq_{\mathbb{Q}_\kappa} p_2$
 - (c) $\text{tr}(p_2) = \text{tr}(p_1)$ and moreover $p_1 \cap {}^{(\alpha_1)} 2 = p_2 \cap {}^{(\alpha_1)} 2$
- (C) the generic of $\mathbb{Q}_\kappa^{\text{am}}$ is $p_\kappa = \cup \{p \cap \alpha : (\alpha, p) \in \mathbb{Q}_\kappa^{\text{am}}\}$.

- Claim 3.29.** 1) p_κ is indeed a generic for $\mathbb{Q}_\kappa^{\text{am}}$.
 2) $\Vdash_{\mathbb{Q}_\kappa^{\text{am}}} "p_\kappa \in \mathbb{Q}_\kappa"$.
 3) If \mathcal{I} is a predense subset of \mathbb{Q}_κ (in \mathbf{V}) then $\Vdash_{\mathbb{Q}_\kappa^{\text{am}}} "if \text{set}(\mathcal{I}) \subseteq \text{set}(\text{nb}(p_\kappa)), \text{ i.e.}$
 4) $\Vdash_{\mathbb{Q}_\kappa^{\text{am}}} "set(\text{nb}(p_\kappa)) \text{ is a member of } \text{id}(\mathbb{Q}_\kappa) \text{ including all the old Borel sets from } \text{id}(\mathbb{Q}_\kappa)"$.

Proof. Easy. □

§ 3(E). Generic and Absoluteness.

- Claim 3.30.** 1) " $p \in \mathbb{Q}_\kappa$ " is a κ -stationary-Borel relation, (see 0.1(4)).
 2) " $p \leq_{\mathbb{Q}_\kappa} q$ ", " $p, q \in \mathbb{Q}_\kappa$ are compatible" are κ -Borel relations (but pedantically there are λ -Borel relations where restrictions to \mathbb{Q}_κ are the above relations).
 3) If κ is weakly compact, then " κ -stationary-Borel" is equivalent to " κ -Borel".
 4) If κ is weakly compact then " $\{p_i : i < \kappa\} \subseteq \mathbb{Q}_\kappa$ is predense" is κ -Borel.

Proof. Straight. □

Similarly

- Claim 3.31.** 1) " $\{p_i : i < \kappa\} \leq \mathbb{Q}_\kappa$ is predense" is $\Sigma_1^1(\kappa)$ this means $\{(i, \eta) : \eta \in p_i, i < \kappa\}$ is $\Sigma_1^1(\kappa)$; similarly for κ -Borel, etc.
 2) $\mathbf{B} \in \text{id}(\mathbb{Q}_\kappa)$ is κ -stationary-Borel relation, where \mathbf{B} is a λ -Borel set.
 3) " $\langle \mathbf{B}_i / \text{id}(\mathbb{Q}_\kappa), i < i(*) \rangle$ is a predense" where $i(*) < \kappa^+$, \mathbf{B}_i is λ -Borel $\text{id}(\mathbb{Q}_\kappa)$; κ -stationary-Borel relations.

Definition 3.32. 1) We say M is a κ -model when:

- (a) $M \subseteq (\mathcal{H}(\kappa)^+, \in)$ is transitive of cardinality κ , $[M]^{<\kappa} \subseteq M$ and M is a model of ZFC^- (i.e. power set axiom omitted)
- (b) similarly for $(\mathcal{H}_{<\kappa^+}(\mathbf{U}), \in)$, \mathbf{U} set of ure-elements.

2) We say η is (M, \mathbb{Q}, η) -generic real when (as in [Sh:630]):

- (a) \mathbb{Q} is a forcing notion definable in M , (absolutely enough in the interesting cases)

- (b) $\eta \in M$ a \mathbb{Q} -name of κ -real, definite by a Borel function from a sequence of κ truth value “ $p \in \mathbf{G}_{\mathbb{Q}}$ ”
- (c) there is $\mathbf{G} \in \mathbb{Q}^M$ generic over M such that $\eta[\mathbf{G}] = \eta$.

Observation 3.33. 1) A κ – Borel set \mathbf{B} belongs to $\text{id}(\mathbb{Q}_{\kappa})$ iff for some κ -real η for every κ -model M to which η belongs we have:

- if ν is (M, \mathbb{Q}_{κ}) -generic real then $\nu \notin B$.

2) If M is a λ -model, $M \models$ “ \mathbb{Q} is $(< \kappa)$ -strategically complete forcing notion (set of class in M sense) (or a definition of sum \mathbb{Q}) and $\mathbf{G} \subseteq \mathbb{Q}$ is generic over M then $M[\mathbf{G}]$ is a κ -model.

We now shall use the forcing from §2 to prove \mathbb{Q}_{κ} is symmetric, i.e. satisfies the generalization of Fubini theorem.

Claim 3.34. Let $\mathbb{Q}_{\kappa} = \mathbb{Q}_{\kappa}''$, i.e. as in §2.

- 1) “ \mathbb{Q}_{κ} -generic” is symmetric, i.e. for κ -model M .
- 2) The following conditions $(*)_1, (*)_2$ on η_1, η_2 are equivalent where

$(*)_{\ell}$ η_{ℓ} is \mathbb{Q}_{κ} -generic over M and $\eta_{3-\ell}$ is \mathbb{Q}_{κ} -generic over $M[\eta]$.

Proof. This follows by the theorem below. □

Theorem 3.35. Assume κ is weakly compact and \mathfrak{x} is a 2-ip with $\lambda_{\mathfrak{x}} = \kappa$ and κ is \mathfrak{x} -weakly compact, so we are in the context of §2. Further assume $\theta_{\alpha, \varepsilon} = 2$ for every α, ε .

If $u \subseteq \mu_{\mathfrak{x}}, w, v$ are disjoint subsets of u and $\Vdash_{\mathbb{Q}_{v, \kappa}}$ “ κ is weakly inaccessible”, then $\Vdash_{\mathbb{Q}_{v, \kappa}^2}$ “ η_w is generic for $(\mathbf{V}[\mathbf{G}_{\mathbb{Q}_{u, \kappa}^2}], \mathbb{Q}_{w, \kappa})$ ”, see Definition 0.5(3), Claim 2.14(6).

Remark 3.36. 1) We can prove this for general $\bar{\theta}$ and \bar{D} (which degenerate when $\bigwedge_{\alpha, \varepsilon} \theta_{\alpha, \varepsilon} = 2$) but we have to restrict the $\bar{D}_{\mathfrak{x}}$ naturally.

2) Will return to this in later parts.

Proof. We use 2.24 freely and recall (by 2.24(3), 2.18)

- ◊₁ $\mathbb{Q}_{v, \kappa} < \mathbb{Q}_{u, \kappa}$ and $\mathbb{Q}_{w, \kappa} < \mathbb{Q}_{u, \kappa}$
- ◊₂ $\mathbb{Q}_{u, \kappa}$ is κ -strategically complete, moreover the strategy in limit stages gives the intersection.

It suffices to prove (B) when (A) holds, where:

- (A) (a) $u \in [\mu]^{\leq \kappa}$
- (b) $v, w \subseteq u$ are disjoint
- (c) $\bar{u} = \langle u_i : i < \kappa \rangle, \bar{v} = \langle v_i : i < \kappa \rangle, \bar{w} = \langle w_i : i < \kappa \rangle$ represent u, v, w respectively
- (d) \mathcal{J} is a $\mathbb{Q}_{v, \kappa}$ -name of a dense open subset of $\mathbb{Q}_{w, \kappa}^1$
- (e) $p_* \in \mathbb{Q}_{u, \kappa}$
- (B) there are p_*, ∂ such that
 - (a) $\mathbb{Q}_{u, \kappa} \models “p_* \leq p_*$ ”
 - (b) $p_* \restriction (v, \partial) \models “\eta_w$ fulfil $\mathcal{J}”$
 - (*)₁ $\mathcal{J}_w = \mathcal{J}$.

Toward this

- (*)₁ choose
 - (a) $\Lambda = \{\text{tr}(q) : q \in \mathcal{I}\}$
 - (b) $\bar{r}_w = \langle r_{\bar{\eta}}^2 : \bar{\eta} \in \Lambda_w^1 \rangle$
 - (c) $\Vdash_{\mathbb{Q}_{v,\kappa}} "r_{\bar{\eta}} \in \mathcal{I} \wedge \text{tr}(r_{\bar{\eta}}) = \bar{\eta} \text{ for every } \bar{\eta} \in \Lambda, \text{ i.e. } r_{\bar{\eta}} \text{ witness } \bar{\eta} \in \Lambda_w^1"$
 - (d) $\mathcal{I}'_1 = \{r_{\bar{\eta}} : \bar{\eta} \in \Lambda_w^1\}$.
- (*)₂ (a) let S_1 be the set of $\partial \in S_{\mathfrak{r}}$ such that $\partial \geq \text{ht}(\text{tr}(p_*))$ and $u_\gamma \cap w = v_\gamma \cap u_\gamma \cap w = w_\gamma$ and $p_* \restriction (v_\partial, \partial) \Vdash_{\mathbb{Q}_{v_\partial, \partial}} "\{r_{\bar{\eta}} \cap \mathbf{T}_{\subseteq w_\partial, \partial} : \bar{\eta} \in \Lambda \cap \mathbf{T}_{\subseteq w_\partial, \gamma}\} \text{ is a } \mathbb{Q}_{v_\partial, \partial}\text{-name of a predense subset of } \mathbb{Q}_{w_\partial, \partial}"$
- (b) for $\partial \in S_1$ let $\Lambda_\partial = \{\bar{\nu} \in \mathbf{T}_{v_\partial, \partial} : \bar{\nu} \in p_* \cap \mathbf{T}_{v_\partial, \partial} \text{ and } p_*^{[\nu]} \text{ forces a value to } \langle r_{\bar{\eta}} \cap \mathbf{T}_{\subseteq w_\partial, \partial} : \bar{\eta} \in \Lambda \cap \mathbf{T}_{\subseteq w_\partial + \partial} \rangle \text{ call it } \langle r_{\bar{\eta}, \bar{\nu}}^\partial : \bar{\eta} \in \Lambda_\partial^\partial \rangle\}$
- (*)₃ S_1 is a stationary subset of S_γ , moreover not in the weakly compact ideal.

[Why? As κ is weakly compact (or \mathfrak{r} -weakly compact if we use \mathfrak{r} as in 1.12, see clause (E) there) and $\Vdash_{\mathbb{Q}_{v,\kappa}} "\{r_{\bar{\eta}} \subseteq \mathbf{T}_{\subseteq w, \kappa} : \bar{\eta} \in \Lambda\} \text{ is a } \mathbb{Q}_{v,\kappa}\text{-name of a predense subset of } \mathbb{Q}_{w,\kappa}"$.]

- (*)₄ there are p_2, S_2 such that
 - (a) $p_2 \in \mathbb{Q}_{v,\kappa}$
 - (b) if $\mathbb{Q}_{v,\kappa} \models "p_2 \leq p'_1"$ then p_* and $(p'_1)^{[+u]}$ are compatible in $\mathbb{Q}_{u,\kappa}$
 - (c) $S_2 \subseteq S_1 \subseteq S_{\mathfrak{r}}$ is stationary moreover
 - (d) if $\partial \in S_1$ then
 - (α) $\text{tr}(p_1) \in \mathbf{T}_{\subseteq w_\partial, \partial}$
 - (β) $u_\partial \cap w = v_\partial$ and $u_\partial \cap w = w_\partial$.

[Why? By §2.]

- (*)₅ let $S_3 = \text{nacc}(S_2)$ and choose $\bar{\rho} \in p_*$ such that $\bar{\rho} \restriction v = \text{tr}(p_2)$
- (*)₆ we define p_{**} as the set of $\bar{\eta} \in \mathbf{T}_{\subseteq u, \kappa}$ such that
 - (a) $\bar{\eta} \leq \bar{\rho}$ or $\bar{\rho} \leq \bar{\eta} \in p_*$
 - (b) if $\partial \in S_3$ and $\text{ht}(\bar{\nu}) < \partial \leq \text{ht}(\bar{\eta})$ then
 - (α) $\bar{\eta} \restriction (v_\partial, \partial) \in \Lambda_\partial$
 - (β) $\bar{\eta} \restriction (w_\partial, \partial) \in \lim_\partial (r_{\bar{\rho}, \bar{\eta} \restriction (v_\partial, \partial)}^\partial)$
- (*)₇ if $\partial \in S_3 \cup \{\kappa\}$ then $p_{**} \cap \mathbf{T}_{\subseteq u_\partial, \partial}$ belongs to $\mathbb{Q}_{u_\partial, \partial}$.

[Why? We prove it by induction on ∂ ; this is straightforward.]

- (*)₈ $\mathbb{Q}_{u,\kappa} \models p_* \subseteq p_{**}$.

[Why? Obvious by the definition.]

- (*)₉ $p_{**} \Vdash "\bar{\eta} \restriction w \text{ fulfill the dense set } \mathcal{I}[\mathbf{G}_{\mathbb{Q}_{u,\kappa}} \cap \mathbb{Q}_{v,\kappa}]"$.

[Why? As the parallel statement holds for every $\partial \in S_3$.]

□3.35

Claim 3.37. *Forcing with \mathbb{Q} add a κ -Cohen real (i.e. a generic for κ -Cohen) when:*

- (a) \mathbb{P} is a forcing notion

- (b) $\Vdash_{\mathbb{P}} \text{"}\eta \in {}^\kappa\kappa \text{ is dominating"}$
- (c) *the above is absolute enough*
- (d) *letting $\mathbb{Q} = \mathbb{Q}_\kappa^{\text{dom}}$ if $M \prec (\mathcal{H}(\chi), \in)$, $\|M\| = \kappa$, $[M]^{<\kappa} \subseteq M$ and $\mathbf{G} \subseteq (\mathbb{Q}_\kappa^{\text{dom}})^M$ is generic over M but $\in \mathbf{V}$ and $\nu = \eta_{\mathbb{Q}_\kappa^{\text{dom}}}[\mathbf{G}]$ then:*
 - $\mathbb{Q}^{M[G]} \subseteq_{\text{ic}} \mathbb{Q}$.

Proof. For $p \in Q$ let $T_p = \{\rho \in {}^{>\kappa}\kappa : p \Vdash \neg(\rho \leq \eta)\}$. □

Remark 3.38. If $\theta_{\alpha, \varepsilon}^{\mathfrak{x}} = 2$ for $\alpha < \mu, \varepsilon < \lambda$; then the $\bar{D}_{\mathfrak{x}}$ disappears.

§ 4. SPECIFIC CASES

Considering §2, we have some free choices, see 2.1, 2.2, ?? mainly the $\bar{D} = \langle D_{\bar{\eta}} : \bar{\eta} \in \mathbf{T}_{\subseteq \mu, \lambda} \rangle$. We shall consider some possible choices and consequences on the properties of $\mathbf{V}^{\mathbb{Q}_{\mu, \lambda}}$. Recall that in all cases

- ⊞ (a) $\mathbb{Q}_{\mu, \lambda}$ is λ^+ -c.c. strategically complete
- (b) $(^\lambda \lambda)^{\mathbf{V}}$ is cofinal in $((^\lambda \lambda)^{\mathbf{V}^{\mathbb{Q}_{\mu, \lambda}}}, \leq_{J_{\lambda}^{\text{bd}}})$.

§ 4(A). Choosing a Parameter and The Cofinality of $\prod_{\varepsilon < \lambda} \theta_\varepsilon$.

Claim 4.1. We define \mathfrak{x} , see Definition 2.1, we choose $\lambda, S_*, \mu > \lambda, \bar{\theta}$ as in 2.3 and we choose $D_{\bar{\eta}}$ as $\{\text{succ}_{\bar{\theta}}(\bar{\eta})\}$.

This \mathfrak{x} is 2-ip indeed, i.e. it satisfies the requirement in 2.1.

Here we use §2

Claim 4.2. Let $\Theta_* \subseteq \{\bar{\theta} : \bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle, \theta_\varepsilon = \text{cf}(\varepsilon) \in (\varepsilon, \lambda)\}$ be non-empty and $\mu > \lambda$.

0) We can choose \mathfrak{x} as 2.1, 2.2 choosing $\theta_{\alpha, \varepsilon}^{\mathfrak{x}} = \text{cf}(\theta_{\alpha, \varepsilon}^{\mathfrak{x}}) \in (\varepsilon, \lambda)$ for $\varepsilon < \lambda, \alpha < \mu$ such that $\bar{\theta} \in \Theta_* \Rightarrow \mu = \sup\{\alpha : \bar{\theta}_\alpha^{\mathfrak{x}} = \bar{\theta}\}$ and for $\nu \in \prod_{\zeta < \varepsilon} \theta_\zeta, \alpha < \lambda$, let $D_{\alpha, \nu} =$

$\{A \subseteq \theta_\varepsilon : A \text{ contains an end segment}\}$.

1) This \mathfrak{x} is a 3-ip and in ??(3), $\Theta_* \subseteq \Theta$.

2) If $u \subseteq \mu, \lambda < \text{cf}(\text{otp}(u)), \bar{\theta} \in \Theta_*$ and $\{\alpha \in u : \bar{\theta}_\alpha = \theta\}$ is unbounded in u then in $\mathbf{V}^{\mathbb{Q}_{u, \lambda}}$ the cofinality of $(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_{\lambda}^{\text{bd}}})$ is $\text{cf}(\text{otp}(u))$.

Proof. 1) Check.

2) By Claim 4.4 below. □_{4.2}

Claim 4.3. 1) If \mathfrak{x} is a 3-ip and $\alpha \in u$ and $\bar{\theta}_\alpha = \bar{\theta}_\alpha^{\mathfrak{x}} \in \Theta$, see ??(3) then $\Vdash_{\mathbb{Q}_{u, \kappa}} \eta_\alpha$ dominate $(\prod_{\varepsilon < \lambda} \theta_{\alpha, \varepsilon})^{\mathbf{V}^{\mathbb{Q}_{u \cap \alpha, \lambda}}}$.

2) Moreover, $\Vdash_{\mathbb{Q}_{u, \kappa}} "(^\lambda 2)^{\mathbf{V}^{\mathbb{Q}_{u \cap \alpha, \lambda}}}$ is meagre".

Proof. 1) Let ρ be a $\mathbb{Q}_{u \cap \alpha, \lambda}$ -name of a member of $\prod_{\varepsilon < \lambda} \theta_{\alpha, \varepsilon}$.

Let $p \in \mathbb{Q}_{u, \lambda}$ so let $p' = p \restriction (\alpha \cap u)$ see the parallel to ??. Let $q \in \mathbb{Q}_{u \cap \alpha, \lambda}$ be as in the parallel to 2.22 for the $\mathbb{Q}_{u \cap \alpha, \lambda}$ -name $\rho(\varepsilon)$ and the condition $p', p \restriction (u \cap \alpha)$ getting $q, \langle (u_\varepsilon, \gamma_\varepsilon) : \varepsilon < \lambda \rangle$ as there. We define a $(u \cap \alpha, u)$ -commitment \mathbf{y} naturally by (check §2, frt?). Now define $r \in \mathbb{Q}_{u, \lambda}$ with domain $\text{Dom}(p) \cap \text{Dom}(q)$ combining p, q and \mathbf{y} .

2) Follows by (1) (recalling $(*)_5$ inside the proof of [Sh:945, 1.3]). □_{4.3}

Claim 4.4. 1) Assume \mathfrak{x} is a 3-ip, $u \subseteq \mu$ and $\text{cf}(\text{otp}(u)) > \lambda$ and $\bar{\theta} \in \Theta$, that is $\{\alpha \in u : \bar{\theta}_\alpha = \theta\}$ is unbounded in u . Then in $\mathbf{V}^{\mathbb{Q}_{u, \lambda}}$ we have $\text{cf}(\prod_{\varepsilon < \kappa} \theta_\varepsilon, <_{J_{\lambda}^{\text{bd}}}) = \text{cf}(\text{otp}(u))$.

2) Moreover $\text{cov}(\text{meagre}) \leq \text{cf}(\text{otp}(u))$ in $\mathbf{V}^{\mathbb{Q}_{u, \lambda}}$.

Proof. 1) $(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}^{\mathbb{Q}_{u, \kappa}}}$ is $\cup \{(\prod_{\varepsilon < \lambda} \theta_\varepsilon)^{\mathbf{V}^{\mathbb{Q}_{u \cap \alpha, \lambda}}} : \alpha \in u \text{ is such that } \bar{\theta}_\alpha = \bar{\theta}\}$ because

- $\text{cf}(\text{otp}(u)) > \lambda$
- $\mathbb{Q}_{u,\kappa}$ satisfies the κ^+ -c.c.
- $\mathbb{Q}_{u,\kappa} = \cup\{\mathbb{Q}_{u\cap\alpha,\kappa} : \alpha \in u, \bar{\theta}_\alpha = \bar{\theta}\}$.

2) By part (j) and 4.3. □_{4.4}

* * *

What about the question from [Sh:945]?

Theorem 4.5. 1) Assume λ is a Laver indestructible supercompact cardinal, (i.e. as in 0.8(1)), $\mu_3 = \mu_3^\lambda \geq \mu_2 \geq \text{cf}(\mu_2) \geq \mu_1 = \text{cf}(\mu_1) > \mu_0 = \text{cf}(\mu_0) = \lambda^+$ and $\mu_2 = \mu_1 \wedge \mu_2 = \mu_2^{<\mu_1}$. Then for some $(< \lambda)$ -strategically complete λ^+ -c.c. forcing notion \mathbb{P} we have $\Vdash_{\mathbb{P}} "2^\lambda = \mu_3, \mathfrak{d}_\lambda = \mu_2, \mathfrak{b}_\lambda = \mu_1 \text{ and } \text{cov}_\lambda(\text{meagre}) = \mu_0"$.
 2) Assuming $\mu_0 = \lambda^+ = 2^\lambda$, (or just $\mu_1 \geq 2^\lambda$ and μ_0 is chosen later). For some choice of $\bar{\theta}$, also in $\mathbf{V}^{\mathbb{P}}$ the cardinal λ is supercompact.

Remark 4.6. Why not $\mu_0 > \lambda^+$? The problem is: why $\text{cov}(\text{meagre}) > \lambda^+$? We intend to deal with it in part II.

Proof. 1) Let $\delta(*) = \mu_3 \times \mu_1$ so $|\delta(*)| = \mu_3, \text{cf}(\delta(*)) = \mu_1$. As in [Sh:945, §1] let $\mathbf{q} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\beta : \alpha \leq \delta(*), \beta < \delta(*) \rangle$ be $(< \lambda)$ -support iteration, \mathbb{Q}_β is chosen as follows: if $\beta < \mu_3$ it is the dominating λ -real forcing $\mathbb{Q}_\lambda^{\text{dom}}$ in $\mathbf{V}^{\mathbb{P}_\beta}$, of course and if $\beta \geq \mu_2$ it is $\mathbb{Q}_{\bar{\theta}}$ in $\mathbf{V}^{\mathbb{P}_\beta}$ ([Sh:945, §1]).

So as there, no cardinal is collapsed and no cofinality changed. Also $\mathbb{P}_{\delta(*)}$ is $(< \lambda)$ -directed complete (see 0.8(1)) λ^+ -c.c.

Also

⊞ $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}_{\delta(*)}}$ satisfies:

(a) $2^\lambda = \mu_2, \mathfrak{d}_\lambda = \mathfrak{b}_\lambda = \mu_1$.

Let \mathfrak{r} be as in 4.2(1) and so as in 2.9 we define $\mathbb{Q}_{\mu_0,\lambda}^2$, for this \mathfrak{r} .

Now force with $\mathbb{Q}_{\mu_0,\lambda}^2$, again it is $(< \lambda)$ -strategically complete λ^+ -c.c., hence so is $\mathbb{P} = \mathbb{P}_{\delta(*)} * \mathbb{Q}_{\mu_0,\lambda}^2$ in \mathbf{V} . By 4.4(2) we have $\mathbf{V}^{\mathbb{P}} \models \text{"cov}(\text{meagre}) \leq \mu_0"$ so equality holds as always $\text{cov}_\lambda(\text{meagre}) > \lambda$. Also as in [Sh:945], $\mathfrak{b}_\lambda = \mu_1 = \mathfrak{d}_\lambda$ and, of course, still $2^\lambda = \mu_2$.

2) Derive \mathbf{V}_1 as in 0.8(2) and let $h_* : \lambda \rightarrow \lambda \cap \text{Card}$ be such that $\mathbf{h}(\alpha) \in \mathcal{H}(h_*(\alpha))$. Clearly $(2^\lambda)^{\mathbf{V}_1} = (2^\lambda)^{\mathbf{V}}$, choose a sequence $\langle \theta_\varepsilon : \varepsilon < \lambda \rangle \in {}^\lambda \lambda$ such that $\theta_\varepsilon = \text{cf}(\theta_\varepsilon) > h_*(\varepsilon) > \varepsilon$: and, if $\mathbf{V} \models 2^\lambda = \lambda^+$ necessarily $\text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_D) = \lambda^+ \neq \mu_0$ if not, still μ_0 as $\text{cf}(\prod_{\varepsilon < \lambda} \theta_\varepsilon, <_{J_\lambda^{\text{bd}}})^{\mathbf{V}_1} \leq (2^\lambda)^{\mathbf{V}_1} = (2^\lambda)^{\mathbf{V}}$. Now in §2, i.e. 4.2, let $\Theta_* = \{\langle \theta_\varepsilon : \varepsilon < \lambda \rangle\}, \theta_{\alpha,\varepsilon} = \theta_\varepsilon$ and use $\mathbb{P}_{\delta(*)}, \mathbb{Q}_{\mu_0,\lambda}, \mathbb{P}$ as above. Let $\mathbf{V}_2 = \mathbf{V}^{\mathbb{P}_{\delta(*)}_1}$ and $\mathbf{V}_3 = \mathbf{V}_2^{\mathbb{Q}_{\mu_0,*}} = \mathbf{V}_1^{\mathbb{P}}$. So the only problem is why $\mathbf{V}_3 \models \text{"}\lambda \text{ is supercompact"}$. To this end use 0.8(2). See more in §5 = 4.7, etc. □_{4.5}

§ 4(B). Large Cardinals.

So we start as in §(4A) with supercompact cardinal and in 4.5(1) end with just strongly inaccessible ones. Can we end up with λ supercompact? Yes, by 4.5(2), 0.8(2). Can we start with a smaller cardinal? For the second we use indescribable cardinals.

Definition 4.7. 1) We say an inaccessible cardinal λ is Π_n^1 -indescribable when for every Π_n^1 sentence ψ (i.e. counting second order quantifiers) if $A \subseteq \mathcal{H}(\lambda)$ and $(\mathcal{H}(\lambda), \in, A) \models \psi$ then for some strongly inaccessible $\lambda_1 < \lambda$, $(\mathcal{H}(\lambda_1), \in, A \cap \lambda_1) \models \psi$.

2) For λ strongly inaccessible and $\chi > \lambda$ we say that λ is χ -indescribable when for every $A \subseteq \lambda$ and first order sentence ψ such that $(\mathcal{H}(\chi), \in, \lambda, A) \models \psi$ for a stationary set of $\lambda_1 < \lambda$ for some $\chi_1 \in (\lambda_1, \lambda)$ we have $(\mathcal{H}(\chi_1), \in, \lambda_1, A \cap \lambda_1) \models \psi$ and $(\mathcal{H}(\lambda_1), \in, A \cap \lambda_1) \prec (\mathcal{H}(\lambda), \in, A)$.

See Magidor-Kanamori [KM78] and or Jech [Jec03]. This version for $n = 1$ is equivalent to “weakly compact”.

Claim 4.8. Let $\mathbf{K} = \{\mathbf{q} : \mathbf{q} = \langle \mathbb{P}_\alpha^*, \mathbb{Q}_\beta^* : \alpha \leq \gamma, \beta < \gamma \rangle \text{ satisfies } \mathbf{q} \in \mathcal{H}(\lambda) \text{ is an Easton support iteration, each } \mathbb{Q}_\beta^* \text{ is } (< |\beta|)\text{-strategically complete}\}$. So (K, \leq) is a λ -complete forcing notion of cardinality λ . Let $\mathbf{G}_K \subseteq \mathbf{K}$ be generic over \mathbf{V} , so really $\mathbf{q}_\lambda = q_\lambda[\mathbf{G}_K]$ is defined naturally, as Easton support iteration of length λ and $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}}[\mathbb{P}_\lambda^*[\mathbf{q}_*]]$.

1) Assume \mathbb{Q} is a forcing notion $\subseteq \mathcal{H}(\lambda^+)^{\mathbf{V}_1}$ definable in $(\mathcal{H}(\lambda^+)^{\mathbf{V}_1}, \in)$ from some parameter and \mathbb{Q} is $(< \lambda)$ -strategically closed and λ^+ -c.c.

(a) If λ is n -indescribable for every n , in \mathbf{V} then so it is in $\mathbf{V}_1^{\mathbb{Q}}$

(b) if ψ is a Σ_{n_2+n} formula and $\mathbf{V} \models \text{“}\lambda \text{ is } n\text{-indescribable”}$ then $\mathbf{V}_1^{\mathbb{Q}} \models \text{“}\lambda \text{ is } n_2\text{-indescribable”}$.

2) If $\chi > \lambda$ and in \mathbf{V} , λ is χ -indescribable, in \mathbf{V}_1 , θ_λ a $(< \lambda)$ -strategical forcing $(\in \mathcal{H}(\chi))$ as in 0.8(2) (or less), then $\Vdash_{\mathbb{Q}} \text{“}\lambda \text{ is } \chi\text{-indescribable”}$.

3) The parallel of §(4A).

Theorem 4.9. 1) In 4.5(1) it is enough that λ is χ -indescribable with appropriate weak-compactness (as required in [Sh:F974, §4], a property which holds in \mathbf{L} if it holds in \mathbf{V}). [But see ??? - 2011.2.18.]

Proof. 1) How we guarantee that $\mathbf{V}^{\mathbb{P}_{\delta(*)}} \models \text{“}\lambda \text{ is weakly compact”}$? We define \mathbf{q} not as there but as in [Sh:F974] applied to $M = (\delta(*), <)$, each \mathbb{Q}_α as above. But is λ weakly compact in $\mathbf{V}^{\mathbb{P}_{\delta(*)}}$?

So let \tilde{T} be a $\mathbb{P}_{\mathbf{q}, \delta(*)}$ -name of a subtree of ${}^\lambda 2$ such that $\alpha < \lambda \Rightarrow 2^\alpha \cap \tilde{T} \neq \emptyset$. So for some $u \in [\delta(*)]^\lambda$, $\Vdash_{\mathbb{P}_{\mathbf{q}, \delta(*)}} \text{“}\tilde{T} \in \mathbf{V}[\langle \eta_\alpha : \alpha \in u \rangle]$ ”, so it is enough to prove $\Vdash_{\mathbb{P}_{\delta(*)}, u} \text{“}\tilde{T} \text{ has a } \lambda\text{-branch”}$. But $\mathbb{P}_{\mathbf{q}, u} \cong \mathbb{P}_{\mathbf{q}, \text{otp}(u)}$. This is definable in $(\mathcal{H}(\lambda^+), \in)$, etc., see [Sh:F974]. That is, without loss of generality $\text{otp}(u) \geq \kappa$. Without loss of generality $\Diamond_{I_\lambda^{\text{wc}}}$ hence there is $\langle <_\kappa : \kappa < \lambda \rangle$ such that $<_\kappa$ is a well ordering of κ such that for every well ordering $<_*$ of $\lambda \setminus \{\kappa : <_\kappa = <_* \restriction \kappa\} \in (I_\lambda^{\text{wc}})^+$. Use this to do a preliminary forcing. $\square_{4.9}$

What about preserving supercompactness?

Theorem 4.10. Assume λ is supercompact, \mathbf{V}_1 is as in 0.8(3). In \mathbf{V}_1 , if $\mu_2 = \mu_2^\lambda \geq \mu_2 = \text{cf}(\mu_1) > \mu_0 = \text{cf}(\mu_0) > \lambda$ then for some $(< \lambda)$ -strategically complete λ^+ -c.c. forcing notion \mathbb{P} in $\mathbf{V}_1^{\mathbb{P}}$ we have (same cardinal and cofinalities and)

- $2^\lambda = \mu_2$
- $\mathfrak{d}_\lambda = \mu_1 = \mathfrak{b}_\lambda$
- $\text{cov}(\text{meagre}) = \mu_0$

- λ is supercompact.

Proof. We first increase 2^λ to μ_2 . Let $\chi > 2^\chi$. Let $\mathbf{j} : \mathbf{V} \rightarrow \mathbf{M}$ be such that $M^\chi \subseteq M$. Then there is a sequence $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle, \theta_\varepsilon \in (\varepsilon, \lambda)$ regular such that $\mathbf{j}(\bar{\theta})_\lambda = \mu_0$. Let \mathbb{P}_χ be a $(< \lambda)$ -support iteration of $\mathbb{Q}^{\text{dom } \bar{\theta}}$ of length μ_0 , so in $\mathbf{V}^{\mathbb{P}_1}$ we have $\text{cf}(\Pi \theta_\varepsilon, <_{J_\lambda^{\text{bd}}}) = \mu_0$ and \mathbb{P}_* is λ^+ -c.c. $(< \lambda)$ -strategically complete. In $\mathbf{V}^{\mathbb{P}_1}$ let $P_{\delta(*)} * \mathbb{Q}_{\mu_0, \lambda}$ be as in the proof of 4.5. The χ -supercompactness is preserved in each step by 0.8(3) by \mathbb{P}_* , and as χ does not depend on \mathbb{P}_* , this holds for every $\chi' > \chi$, so \mathbb{P}_* preserves supercompactness of λ_1 . Similarly in later stages. $\square_{4.10}$

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